An analogue to digital converter quantizes an input voltage which is randomly distributed, with a standard deviation very much larger than $\Delta V$, the spacing of the digitised levels. Show that the r. m. s. of the difference between the input and the digitised output is $\Delta V/\sqrt{12}$.

In Experimental Methods lecture 8, slide 9, the case of Gaussian errors is considered. I thought I would add some detail to make the effect of the choice of noise model more obvious.

The case considered on the slide is a measured data set $\{x_i, y_i\}_{i=1}^N$ with no error in $x$ and Gaussian error with known $\sigma_i$ in $y$, sampled from a true signal $y(x)$.

Reminder of the Gaussian distribution:

$$N(x|\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

By using a Gaussian noise model, we are saying that each measured value $y_i$ is a sample from a Gaussian whose mean is the true value $y(x_i)$

$$p(y_i|x_i, y, \sigma_i) = N(y_i|y(x_i), \sigma_i) \, .$$

Because the Gaussian is just translated by a change in mean, we can express the same thing differently by saying that each measured value $y_i$ is the true value $y(x_i)$ to which Gaussian noise $\delta_i$ has been added

$$y_i = y(x_i) + \delta_i$$

or

$$p(y_i - y(x_i)|x_i, y, \sigma_i) = N(0, \sigma_i) = p(\delta_i|\sigma_i) \, .$$

The difference between the true signal $y(x_i)$ and the measurement $y_i$ is given by $\delta_i$ with distribution $N(0, \sigma_i)$, and the r. m. s. value of the difference for a single measurement is $\sqrt{\langle \delta^2 \rangle}$.

$$\langle \delta^2 \rangle = \int_{-\infty}^{\infty} \delta^2 p(\delta) \, d\delta$$

$$= \int_{-\infty}^{\infty} \delta^2 N(0, \sigma) \, d\delta$$

$$= \int_{-\infty}^{\infty} \delta^2 \frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{\delta^2}{2\sigma^2}\right) \, d\delta$$

Back to our favourite integral from the probability densities of hydrogen, e. g., Quantum Mechanics problem sheet 3, question 35:

$$I_n = \int_{-\infty}^{\infty} x^n e^{-ax^2} \, dx$$

$$= \int_{-\infty}^{\infty} x^{n-1} xe^{-ax^2} \, dx$$

$$= \left[ x^{n-1} e^{-ax^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (n-1)x^{n-2} e^{-ax^2} \, dx$$

$$= \frac{n-1}{2a} I_{n-2}$$
\[ I_0 = \int_{-\infty}^{\infty} e^{-\alpha x^2} \, dx \]
\[ I_0^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha (x^2+y^2)} \, dx \, dy \]
\[ = \int_0^{2\pi} d\theta \int_0^{\infty} dr \, r e^{-\alpha r^2} \]
\[ = \left[ \frac{\theta^2 \pi}{\alpha} \right]^\infty_0 = \frac{\pi}{\alpha} \]
\[ I_0 = \sqrt{\frac{\pi}{\alpha}} \]

\( I_1 = 0 \) as it is the integral of an odd function.
\[
\langle \delta^2 \rangle = \frac{1}{I_0} \frac{2 - 1}{2\alpha} I_0
\]
\[ = \sigma^2 \]
\[ \sqrt{\langle \delta^2 \rangle} = \sigma \]

Hopefully you would have expected the r.m.s. value of a variable whose distribution is given by a Gaussian of zero mean and width \( \sigma \) to be \( \sigma \)!

Now for the ADC quantization problem. The effect of quantization is to add an error to the signal whose magnitude is at worst equal to half the spacing \( \ell = \Delta V \) of the digitised levels. The simplest consistent noise model is a uniform distribution over the interval \( \pm \ell/2 \):

\[
p(\delta|\ell) = \begin{cases} \frac{1}{\ell} & \text{for } |\delta| \leq \ell/2 \\ 0 & \text{otherwise.} \end{cases}
\]
\[
\langle \delta^2 \rangle = \int_{-\infty}^{\infty} \delta^2 \, p(\delta) \, d\delta
\]
\[ = \int_{-\ell/2}^{\ell/2} \delta^2 \, \frac{1}{\ell} \, d\delta
\]
\[ = \frac{1}{\ell} \left[ \frac{\delta^3}{3} \right]_{-\ell/2}^{\ell/2}
\]
\[ = \frac{\ell^2}{12}
\]
\[ \sqrt{\langle \delta^2 \rangle} = \frac{\ell}{\sqrt{12}} \]

So the r.m.s. of the difference between the input and the digitised output is \( \Delta V / \sqrt{12} \).