

## Ph103b: Solutions to Problem Set 4

**Problem 1.** A flat bottomed bowl of radius  $R$  is filled with water to depth  $H$ .

- a) Provide an approximate expression for the period,  $P$ , of the sloshing mode.
- b) Estimate the damping timescale,  $T_\nu$ , in the small amplitude limit.
- c) Evaluate  $P$  and  $T_\nu$  for  $R = 10$  cm and  $H = 5$  cm.

**a)** Presumably the bowl is shallow (wider than it is deep). The sloshing mode is a long wavelength mode: it is a gravity wave on shallow water. Therefore  $\omega^2 = gHk^2$ , so  $v_{\text{ph}} = v_g = (gH)^{1/2}$ . The wave has a round-trip travel distance of  $\sim 4R$ , so the period is

$$P \sim \frac{4R}{v_g} \sim \frac{4R}{(gH)^{1/2}}. \quad (1.1)$$

**b)** The kinetic energy in the sloshing dissipates in the viscous boundary layer at the bottom of the bowl. The damping time is the ratio of the kinetic energy to the dissipation rate. The kinetic energy lives mostly in the sideways velocity of the water (for shallow bowls). The shallow water is in hydrostatic equilibrium, so this velocity is independent of depth, except in the boundary layer.

This velocity, call it  $v_s$ , goes to zero near the bottom, in a thin boundary layer. The boundary layer doesn't grow for the full period, but rather for a characteristic time,  $t \sim 1/\omega = P/2\pi$ , so the boundary layer has thickness

$$\delta \sim (\nu/\omega)^{1/2}. \quad (1.2)$$

The viscous stress is  $\rho\nu\nabla\mathbf{v} \sim \rho\nu v_s/\delta$ . Multiplying by the area of the bottom,  $A \sim R^2$ , and the velocity,  $v_s$ , we get the dissipation rate,

$$P_d \sim \rho\nu v_s^2 R^2/\delta. \quad (1.3)$$

The kinetic energy in the flow is

$$E \sim Mv_s^2 \sim \rho R^2 H v_s^2. \quad (1.4)$$

The damping timescale is then

$$T_\nu \sim \frac{E}{P_d} \sim \frac{\rho R^2 H v_s^2}{\rho\nu R^2 v_s^2/\delta} = \frac{H\delta}{\nu} \quad (1.5)$$

From 1.2, we have  $\delta \sim (\nu/\omega)^{1/2}$ , so

$$T_\nu \sim \frac{H(\nu/\omega)^{1/2}}{\nu} \sim \left( \frac{H^2}{\nu} \times \frac{1}{\omega} \right)^{1/2}. \quad (1.6)$$

The first quantity,  $H^2/\nu$ , is the diffusion time from the bottom to the top of the bowl, and  $1/\omega$  is the oscillation timescale. So  $T_\nu$  is the geometric mean of the diffusion time and the oscillation period. When dissipation is confined to a thin boundary layer, the damping time is usually the geometric mean of a dissipative time and a dynamical time (as it was for the teacup despinning example treated in class). If we use 1.1 to substitute for  $1/\omega$ , we get (neglecting all constants of course)

$$T_\nu \sim \left( \frac{R^2 H^3}{g\nu^2} \right)^{1/4}. \quad (1.7)$$

c) Putting  $R = 10 \text{ cm}$  and  $H = 5 \text{ cm}$  into 1.1, we get

$$P \sim \frac{4 \times 10 \text{ cm}}{(1000 \text{ cm s}^{-2} \times 5 \text{ cm})^{1/2}} \sim 0.6 \text{ s.}$$

From 1.7 we get

$$T_\nu \sim \left( \frac{100 \text{ cm}^2 \times 125 \text{ cm}^3}{1000 \text{ cm s}^{-2} \times 10^{-4} \text{ cm}^4 \text{ s}^{-2}} \right)^{1/4} \sim 20 \text{ s.} \quad (1.8)$$

One of the important timescales of the El Niño phenomenon is the period of the sloshing mode of the *thermocline*. The thermocline can be thought of as the interface between the warmer surface waters in the top 100–200 meters of the ocean and the colder deep waters.

Because of buoyancy the effective gravity is reduced and is given by

$$g' = \frac{\rho_{\text{warm}} - \rho_{\text{cold}}}{\rho_{\text{cold}}} g = \alpha(\Delta T)g, \quad (1.9)$$

where  $\alpha$  is the thermal expansivity of sea water. We will take the length of the Pacific Ocean to be  $L \sim 20,000 \text{ km} = 2 \cdot 10^9 \text{ cm}$ . Then the sloshing period for the Pacific Ocean thermocline is

$$P \sim \frac{2L}{[\alpha(\Delta T)gH]^{1/2}} \approx \frac{2 \times 2 \cdot 10^9 \text{ cm}}{(10^{-4} \text{ K}^{-1} \times 20 \text{ K} \times 10^3 \text{ cm s}^{-2} \times 2 \cdot 10^4 \text{ cm})^{1/2}} \sim 8 \text{ months.} \quad (1.10)$$

**Problem 2.** *Shabu shabu*, which translates as “swish swish”, is a method of Japanese cooking in which thin pieces of meat are moved through near boiling water. How does the heat flux into the meat vary with the speed at which it is moved?

Moving thin pieces of meat through hot water increases the rate at which they cook by decreasing the thickness of the thermal boundary layer. This effect is significant provided the thickness of the meat is not much greater than that of the boundary layer. The viscous boundary layer thickness is

$$\delta_\nu \sim (\nu t)^{1/2}, \quad (1.11)$$

where  $t$  is the time taken to disrupt the boundary layer. For a flat piece of meat of length  $l$  moving at velocity  $v$  (clearly an idealization),  $t \sim l/v$ . So the heat flux into the meat varies as  $v^{1/2}$  for sufficiently thin pieces of meat. How thin is sufficiently thin? For  $t \approx 1 \text{ s}$ ,  $\delta_\nu \sim 0.1 \text{ cm}$ .

A more accurate calculation must take into account that heat diffuses in water more slowly than momentum. The ratio of the diffusivities is known as the Prandtl number,  $Pr \equiv \nu/\kappa \approx 6$  for water. As shown below, the thermal boundary layer is thinner than the viscous boundary layer by a factor  $Pr^{-1/3} \approx 0.6$  for water.

What about the actual cooking time? Probably a few seconds will do provided the meat, which is frozen when sliced, has thawed before being cooked.

We might rashly expect—by analogy with the viscous boundary layer thickness—that

$$\delta_t \sim \left( \frac{\kappa l}{v} \right)^{1/2}, \quad (1.12)$$

so that the ratio of viscous to thermal boundary layer thicknesses would be

$$f \equiv \frac{\delta_\nu}{\delta_t} \sim \left( \frac{\nu l/v}{\kappa l/v} \right)^{1/2} \sim Pr^{1/2}. \quad (1.13)$$

But this result is slightly wrong; we now fix it up. We will assume the velocity profile in the viscous boundary layer is linear, going from 0 to  $v$  in a thickness  $\delta_\nu$  (this assumption is reasonable for a flat plate, and an exact calculation of the velocity profile, known as the *Blasius profile*, bears us out). From 1.13, the thermal boundary layer lives inside the viscous one (for  $Pr > 1$ ). At the thermal boundary layer, the flow velocity is  $v/f$  (because of the linear velocity profile in the viscous boundary layer). Therefore in 1.13 we replace  $\kappa l/v$  with  $f\kappa l/v$ , to get an implicit equation for  $f$ :

$$f \sim \frac{Pr^{1/2}}{f^{1/2}}. \quad (1.14)$$

Solving for  $f$ , we find

$$f \equiv \frac{\delta_\nu}{\delta_t} \sim Pr^{1/3}, \quad (1.15)$$

which is the third root we claimed above.

**Problem 3.** *At what distance would an ice cube in circular orbit around the sun just sublime in the age of the solar system? The vapor pressure of water over ice is*

$$\log_{10} \left( \frac{P}{\text{dyne cm}^{-2}} \right) \approx 13.5 - \frac{2.67 \times 10^3}{T},$$

where  $T$  is expressed in degrees Kelvin.

We will first calculate the required vapor pressure to sublime in the required time. From the pressure we will use

$$\log_{10} \left( \frac{P}{\text{dyn/cm}^2} \right) = 13.5 - \frac{2670}{(T/^\circ\text{K})} \quad (1.16)$$

to calculate the required ice temperature, and from that the orbit distance.

The vapor pressure is  $P = nkT$ , where  $n$  is the number density of water molecules in the vapor. This pressure corresponds to an incoming flux,  $F \sim nv/6 = Pv/6kT$  (assuming each molecule sticks when it hits). The magic 6 is because only one of the ‘six’ possible directions is towards the surface. In equilibrium, this incoming flux equals the outgoing flux of water molecules breaking free. In our problem, we have no incoming flux (the water vapor escapes into a vacuum and goes to infinity), but the outgoing flux is still  $F \sim Pv/6kT$ . Then the mass flux leaving all six surfaces is

$$\dot{M} \sim 6L^2 m \frac{Pv}{6kT} \sim PL^2 \left( \frac{3m}{kT} \right)^{1/2}, \quad (1.17)$$

where  $m$  is the mass of a water molecule,  $L$  is the side length, and we’ve used the thermal velocity,  $v \sim (3kT/m)^{1/2}$ . With  $M \sim \rho L^3$ , we have

$$\frac{\dot{M}}{M} \sim \frac{P}{\rho L} \sqrt{\frac{3m}{kT}}, \quad (1.18)$$

The timescale is  $\tau \sim M/\dot{M}$ . So the required vapor pressure is

$$P \sim \frac{\rho L c}{\tau} \sqrt{\frac{kT}{3m}} = \frac{\rho L c}{\tau} \sqrt{\frac{kT}{3mc^2}}. \quad (1.19)$$

We will put in  $mc^2 \sim 18 \text{ GeV}$ ;  $L = 2 \text{ cm}$ ; and  $\tau \sim 5 \cdot 10^9 \text{ years}$ . We want to evaluate  $P$  using 1.19, which will tell us  $T$  via 1.16. But 1.19 involves the unknown temperature. We cut the Gordian knot by picking some reasonable  $T$  to plug into 1.19. The resulting error in the pressure will cause only a small error in the final temperature because in 1.16, the vapor pressure varies very rapidly with temperature. We will guess  $T = 50 \text{ K}$ , or  $kT \sim 4 \text{ meV}$ , to plug into 1.19. Then

$$P \sim \frac{1 \text{ g cm}^{-3} \times 2 \text{ cm} \times 3 \cdot 10^{10} \text{ cm s}^{-1}}{5 \cdot 10^9 \text{ years} \times 3 \cdot 10^7 \text{ sec/year}} \times \left( \frac{0.004 \text{ eV}}{3 \times 18 \cdot 10^9 \text{ eV}} \right)^{1/2} \sim 10^{-13} \text{ dyn/cm}^2. \quad (1.20)$$

Putting  $P \sim 10^{-13} \text{ dyn/cm}^2$  into 1.16, we find  $T \approx 100 \text{ K}$ . If we had guessed this value of  $T$  originally in 1.19, the resulting  $T$  here would have been 101 K, which is a small difference, as we hoped.

This temperature corresponds to a blackbody power of

$$P \sim 6L^2\sigma T^4 \sim 6 \times 4 \text{ cm}^2 \times 6 \cdot 10^{-5} \frac{\text{erg}}{\text{cm}^2 \text{ s K}^4} \times 10^8 \text{ K}^4 \sim 1.5 \cdot 10^5 \text{ erg/sec}.$$

From Purcell's sheet, the sun puts out an energy  $P_{\text{sun}} \sim 4 \cdot 10^{33} \text{ erg/sec}$ . The amount absorbed by the ice cube at radius  $R$  is  $P_{\text{in}} \sim f P_{\text{sun}} L^2 / 4\pi R^2$ , where  $f = 1 - A \sim 0.5$  is a fudge factor for the albedo of ice. Equating this incident flux to the radiated flux, we find the orbital radius is

$$R \sim \left( \frac{f L^2 P_{\text{sun}}}{4\pi P} \right)^{1/2} \sim \left( \frac{0.5 \times 4 \text{ cm}^2 \times 4 \cdot 10^{33} \text{ erg s}^{-1}}{4 \times 3 \times 1.5 \cdot 10^5 \text{ erg/sec}} \right)^{1/2} \sim 7 \cdot 10^{13} \text{ cm} = 7 \cdot 10^8 \text{ km}. \quad (1.21)$$

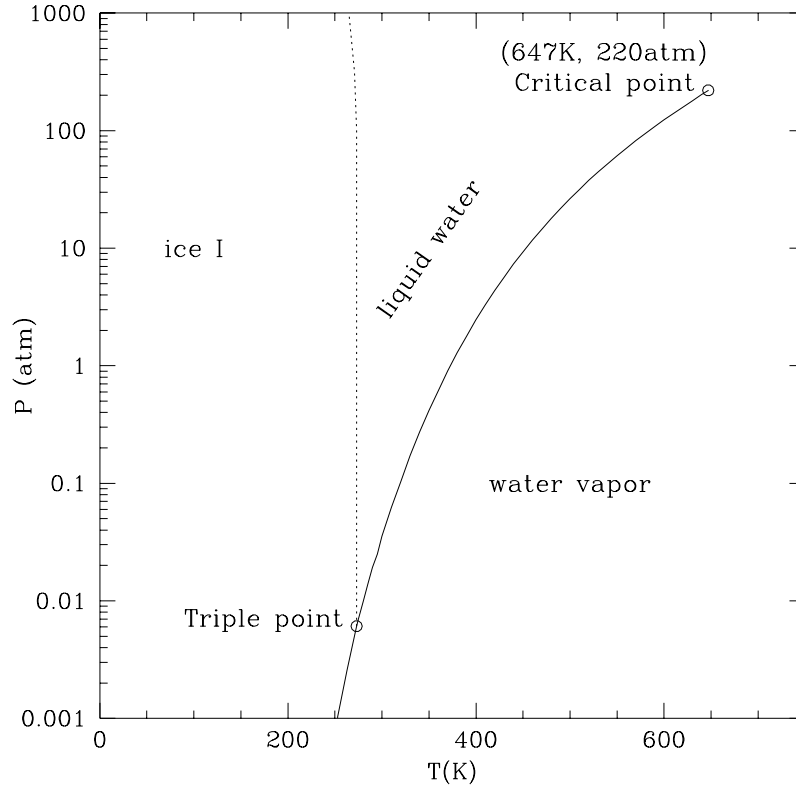
The ice cube is roughly at Jupiter.

**Problem 4.** *The future of water on earth. Care and thought, particularly in parts (d) and (e), will be rewarded.*

- Estimate the total mass of water in the atmosphere at any given time.*
- Estimate the total mass of water in the oceans.*
- How many years' worth of the solar energy flux on earth would be required to completely evaporate the oceans? Why don't they evaporate?*
- To what temperature would the surface of the earth have rise so that there was no longer liquid water on the surface of the earth? The phase diagram on the next page may be useful.*
- By how much must the luminosity of the sun increase in order to reach the state you found in (d)? Five billion years hence, the luminosity of the sun will begin to increase, eventually becoming  $10^3$  times more luminous than it is now.*

**a)** From class, the saturation density of water at  $T = 20^\circ \text{C}$  is  $\rho_{\text{sat}} \sim 1.5 \cdot 10^{-5} \text{ g cm}^{-3}$ , which applies to say the first 3 km of the atmosphere (where  $T \sim 20^\circ \text{C}$ ); the water content falls exponentially with height because the temperature falls linearly, so we can pretend all the water is in this lower part of the atmosphere. Then with  $R \sim 6 \cdot 10^8 \text{ cm}$  and  $H \sim 3 \cdot 10^5 \text{ cm}$ , we have

$$M_{\text{H}_2\text{O}} \sim 4\pi R^2 H \rho_{\text{sat}} \sim 4 \times 3 \times 4 \cdot 10^{17} \text{ cm}^2 \times 3 \cdot 10^5 \text{ cm} \times 1.5 \cdot 10^{-5} \text{ g cm}^{-3} \sim 2 \cdot 10^{19} \text{ g}. \quad (1.22)$$



**Figure [4b]:** Phase diagram of  $\text{H}_2\text{O}$ .  $1 \text{ atm} \simeq 10^6 \text{ dyne cm}^{-2} = 10^5 \text{ pascal}$ .

**b)** The oceans have an average depth  $d \sim 2.7 \text{ km} = 2.7 \cdot 10^5 \text{ cm}$ . [The *Smithsonian Physical Tables*, p. 773, gives the average depth as  $\sim 3.8 \text{ km}$ . Multiplying by the coverage factor of 70% gives  $d \sim 2.7 \text{ km}$ .]. So

$$M_{\text{ocean}} \sim 4\pi\rho R^2 d \sim 4 \times 3 \times 1 \text{ g cm}^{-3} \times 4 \cdot 10^{17} \text{ cm}^2 \times 2.7 \cdot 10^5 \text{ cm} \sim 1.3 \cdot 10^{24} \text{ g}. \quad (1.23)$$

**c)** The heat of vaporization of water is  $L_{\text{vap}} \sim 600 \text{ cal cm}^{-3}$  which is  $2.5 \cdot 10^{10} \text{ erg cm}^{-3}$ . To evaporate a  $1 \text{ cm}^2 \times 2.7 \text{ km}$  column of ocean will take  $2.5 \cdot 10^{10} \text{ erg cm}^{-3} \times 2.7 \cdot 10^5 \text{ cm}^3 \sim 7 \cdot 10^{15} \text{ erg}$ . The average solar flux onto  $1 \text{ cm}^2$  of the earth is  $2 \cdot 10^5 \text{ erg s}^{-1}$ , so the time to evaporate the oceans, assuming no rain, is

$$\tau \sim \frac{7 \cdot 10^{15} \text{ erg}}{2 \cdot 10^5 \text{ erg s}^{-1}} \sim 3.5 \cdot 10^{10} \text{ s} \sim 1000 \text{ years}. \quad (1.24)$$

So the oceans should be long gone by now.

The oceans do not evaporate away because water vapor condensing in the atmosphere gives up its latent heat to the air, causing the air to become more buoyant. The resulting convection carries the energy which went into vaporizing water into the upper troposphere where it is radiated back to space.

**d)** If all the water in the Earth's oceans were to be vaporized, there would be a (vaporized)  $d = 2.7 \text{ km}$ -high column of liquid water over the whole surface of the earth (i.e. the oceans would

just get elevated and vaporized). The atmospheric pressure at the surface would be the weight of water over the surface (the air is negligible here), so

$$P = \rho g d \sim 1 \text{ g cm}^{-3} \times 1000 \text{ cm s}^{-2} \times 2.7 \cdot 10^5 \text{ cm} \sim 270 \cdot 10^6 \text{ dyn/cm}^2 = 270 \text{ atm.} \quad (1.25)$$

This pressure is above the critical pressure of 220 atm. Above the critical point, there is no difference between liquid and vapor states (at the critical point, vapor is compressed to the same density,  $0.3 \text{ g cm}^{-3}$  as the liquid, and there are large fluctuating regions of liquid and vapor as the critical point is approached from below). As the earth's temperature rises the vapor will move along the equilibrium curve (solid line in the graph). Just below 647K, most of the water would be in the vapor state. Once the last bits were vaporised, the temperature would no longer matter as long as it stayed above 647K. Of course the heating of the atmosphere might send some of the water (30% would suffice) into space, reducing the pressure below the critical point, so that the atmosphere would be a steamy but recognisable vapor just below 647K.

e) The ratio of this temperature to the present temperature is  $\sim 647/270 \sim 2.4$ . Since fluxes scale as the fourth power of the temperature, the luminosity must increase by a factor of  $2.4^4 \sim 30$ . The solar luminosity must therefore increase by a factor of 30. This result is debatable because it assumes the albedo of the earth stays constant—which it won't because the resulting clouds will reflect much more light—and it neglects the greenhouse effect, which will have the opposite effect, of keeping the surface very hot. And who knows which factor wins?

Furthermore, once the critical temperature is reached, critical point opalescence will lead to an almost opaque atmosphere. However the geothermal heat flux ( $\sim 0.1 \text{ W m}^{-2}$ ) will create a very small temperature gradient in the opaque atmosphere, but the atmosphere will be more or less opaque and isothermal.

**Problem 5.** *How much warmer is a big city [say  $10^7$  people in a square 20km on a side] than the surrounding countryside? (Hint: the average American uses 10kWatt). Treat two cases:*

- a) *The city is trapped under a breezeless inversion layer, so all heat must be radiated.*
- b) *The heat is convected up into the atmosphere and carried away by horizontal winds.*

a) The power produced in the city, which is mostly radiated [as an exercise: estimate the heat flux into the earth.], is

$$P \sim 10^7 \text{ people} \times 10^{11} \text{ erg s}^{-1} \text{ person}^{-1} = 10^{18} \text{ erg s}^{-1}. \quad (1.26)$$

Per unit area,

$$F = \frac{P}{A} \sim \frac{10^{18} \text{ erg}}{(2 \cdot 10^6 \text{ cm})^2} \sim 3 \times 10^5 \text{ erg s}^{-1} \text{ cm}^{-2}. \quad (1.27)$$

If this power is radiated,

$$F \sim \sigma T_{\text{city}}^4 - \sigma T_{\text{country}}^4. \quad (1.28)$$

If  $T_{\text{country}} \approx 300 \text{ K}$ , then the difference in temperature will be  $T_{\text{city}} - T_{\text{country}} \approx 40 \text{ K}$ !

b) Assuming that the wind blows at 20 km/hr, the wind will take an hour to cross the city. Therefore there will be a city-hour of heat in the city at all times. So  $\Delta E = 10^{18} \text{ erg/s} \times 3600 \text{ s} \sim 4 \cdot 10^{21} \text{ erg}$ . To find the temperature difference, we note that  $\Delta E \sim (7/2) N k \Delta T$ , where  $N$ , the number of particles,

is  $N = \rho V / (28 \times 1.7 \cdot 10^{-24} \text{ g})$ . If the heat is mixed by convection to a height of 1 km, the volume is  $V \sim 400 \text{ km}^3 = 4 \cdot 10^{17} \text{ cm}^3$ . With a density of  $\rho \sim 10^{-3} \text{ g cm}^{-3}$ , the temperature rise would be about 1 K. Urban “heat islands” are in fact 1 – 4 K warmer than the countryside. Get out of New York City in summer, especially on days without breezes!