

**1999**

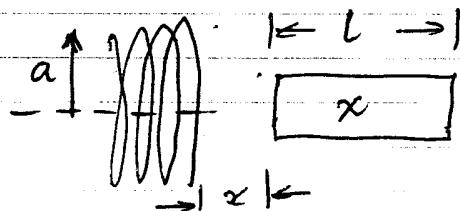
*Paper 1*

## A1 - Disconnect electromagnet

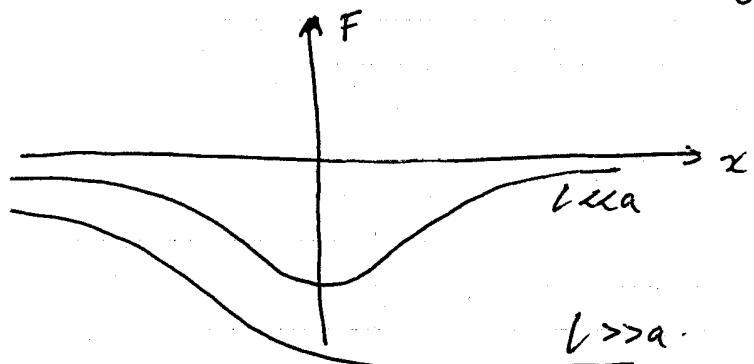
- Faraday :  $E = - \frac{d}{dt} \Phi$
- Turn on : back emf, opposes applied voltage  
current rises steadily
- Turn off : field collapses  
high  $\frac{d\Phi}{dt}$   
high voltage, gives arc
- Possible math for  $E = \pm L \frac{di}{dt}$ .
- No maths for remanent magnetism, hysteresis.

## A2 - Paramagnetism

- induced magnetic dipole  $m = \chi H$

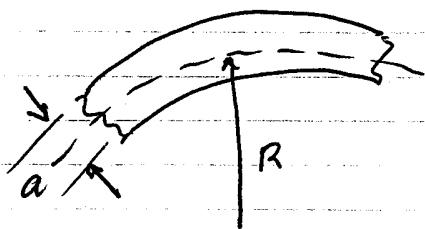


- $F = -m_0 \nabla B$  (m-gad)



- ( • Field is solenoid uniform, changes near end  
• scale of curve must be  $\sim a$  )

### A3 — Bending beam problem



$$e = \frac{\delta l}{l} = \frac{a/2}{R}$$

$$\sigma_c = F/A \quad (= \frac{6N}{\pi(1m)^2})$$

$$\sigma = cY$$

$$\text{Hence } R = \frac{a/2 \cdot YA}{F}$$

$\sigma$  = Stress

$Y$  = Young modulus

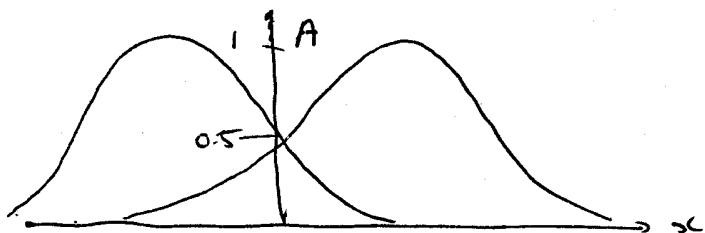
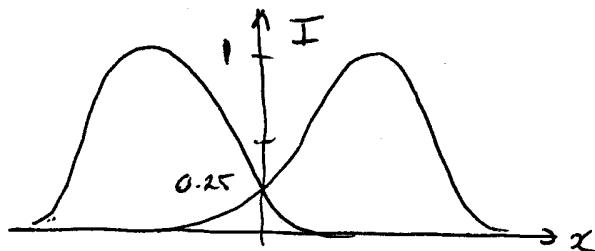
$e$  = strain.

= . 5mm

### A4 — interference

- coherent — add amplitudes — "Sparrow" criterion
- incoherent — add intensities — "Rayleigh" " Taylor"

Cross over at  $\frac{1}{2}$ -amplitude is further than  $\frac{1}{2}$  intensity.



A5 - Visibility plot,  $V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = 8$

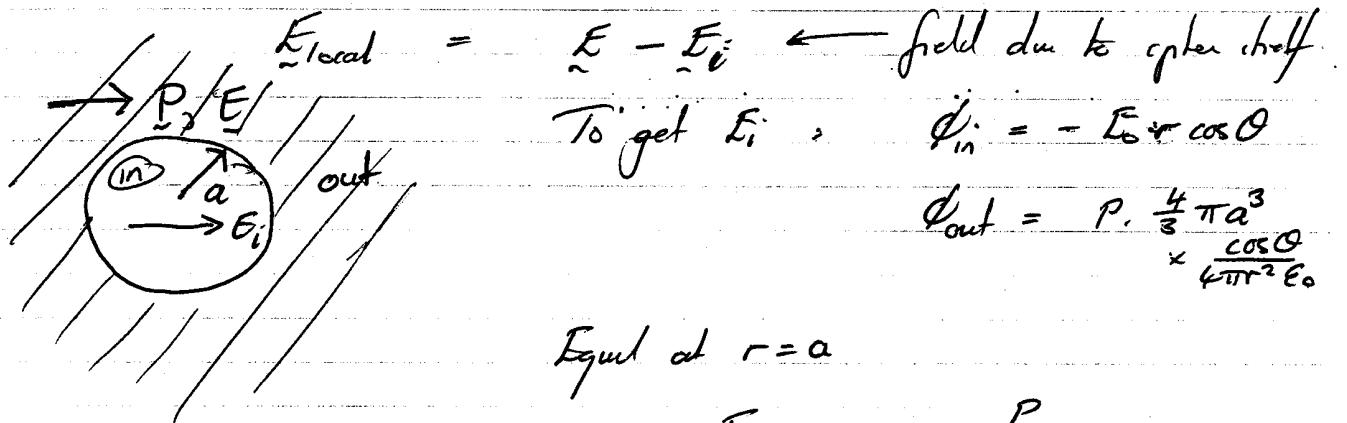
- not zero visibility  $\Rightarrow$  coherent
- fringe visibility = FJ of spectrum
- Single subharmonic max  $\Rightarrow$  3 equal peaks
- slow decay  $\Rightarrow$  finite (non-zero) line width

B6 : Clausius-Mosotti (not in 2003/4 course)

a) Bookwork. microscopic :  $P = N\rho = N\alpha E_{\text{local}}$

macroscopic :  $P = (\epsilon_r - 1)\epsilon_0 E$

note :  $E_{\text{local}} \neq E$



Equal at  $r=a$

$$\Rightarrow E_i = -\frac{P}{3\epsilon_0}$$

$$\Rightarrow E_{\text{local}} = E + \frac{P}{3\epsilon_0}$$

Back to stat

$$P = N\alpha (E + \frac{P}{3\epsilon_0})$$

$$(\epsilon_r - 1)\epsilon_0 E = N\alpha (E + \frac{(\epsilon_r - 1)E}{3})$$

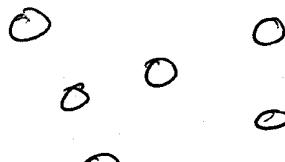
$$\alpha = \frac{3\epsilon_0}{N} \left( \frac{\epsilon_r - 1}{\epsilon_r + 2} \right) *$$

Important? Relates micro to macro properties, shows molecular interactions.

b) Isolated conducting sphere,  $P = 4\pi\epsilon_0 a^3 E = \alpha E$



$$\text{Hence } \alpha = 4\pi\epsilon_0 a^3$$



$$\text{and } f = \frac{4}{3}\pi a^3 N$$

$$\text{From } * \quad \frac{3\epsilon_0}{N} \left( \frac{\epsilon_r - 1}{\epsilon_r + 2} \right) = 4\pi\epsilon_0 a^3$$

(random)

$$\epsilon_{\text{eff}} = \frac{1+2f}{1-f} //$$

## B7 — Method of Images

$$a) \quad \vec{E} = -\nabla \phi \Rightarrow \nabla \cdot \vec{E} = \rho_{\text{free}}/\epsilon \Rightarrow \nabla^2 \phi = \rho_{\text{free}}/\epsilon$$

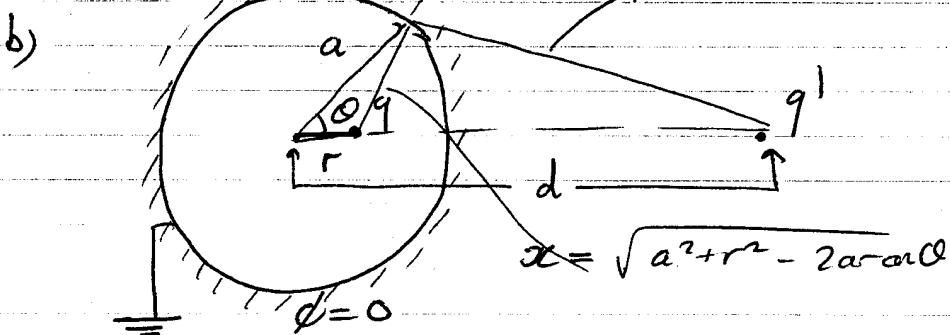
If no free charges,  $\nabla^2\phi = 0$  (Laplace)

Cheys conquerors : hence our great solution

Must satisfy boundary conditions.

Method of images makes plunger also choose so as to satisfy all the boundary conditions without the conductive/insulating boundaries. Then calculation in central region must be identical, but may be more easily calculated.

$$x' = \sqrt{d^2 + a^2 - 2ad \cos \theta}$$



$$x = \sqrt{a^2 + r^2 - 2ar \cos \theta}$$

To satisfy  $\phi = 0$  at an sphere,  $\nabla \phi$

$$\phi = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{x} + \frac{q'}{x'} \right) = 0$$

$$\Rightarrow \frac{g/g'}{-} = \frac{x/x'}{x'}$$

$$\text{So } d^2 + a^2 - 2ad \cos\theta = -\frac{q^2}{q} (a^2 + r^2 - 2ar \cos\theta) + 0$$

$$\Rightarrow \text{d} = a^2/c$$

$$\text{and } \text{ii) } g' = -\left(\frac{d}{dx}\right)g \quad \} \text{ algebra.}$$

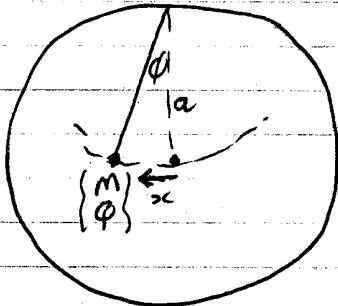
$$\text{Force : } F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} = \frac{q q'}{4\pi\epsilon_0 (d-r)^2}$$

$$= \frac{q^2 ar}{4\pi\epsilon_0 (a^2 - r^2)^2} //$$

c) See over

B7 (ctd)

e)

if  $\phi = 0$ , simple pendulum,

$$F = -mg \sin \phi = -mg \frac{x}{a}.$$

if  $\phi \neq 0$ , add ~~for~~ electric force

$$F = -\frac{mgx}{a} + \frac{Q^2ax}{4\pi\epsilon_0(a^2+x^2)^2}$$

Unstable equilibrium if  $\left. \frac{\partial F}{\partial x} \right|_{x=0} > 0$ 

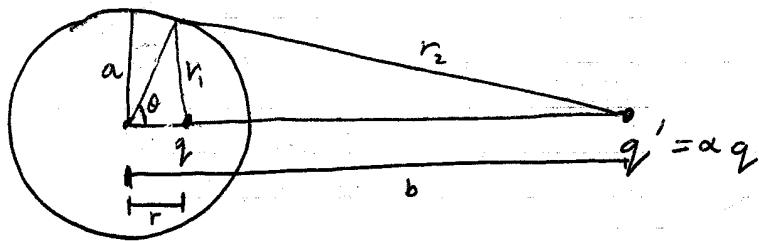
$$\begin{aligned} \frac{\partial F}{\partial x} &= -\frac{mg}{a} + \frac{Q^2a}{4\pi\epsilon_0(a^2+x^2)^2} \Big|_{x=0} + \left\{ \text{Term} = 0 \text{ at } x=0 \right\} \\ &= -\frac{mg}{a} + \frac{Q^2a}{4\pi\epsilon_0 a^3} \end{aligned}$$

Equal to 0 when

$$Q^2 = 4\pi\epsilon_0 mg a^2$$

1999 I 7 (i) Bookwork.

(i)



$$r_1^2 = a^2 + r^2 - 2ar \cos\theta$$

$$r_2^2 = a^2 + b^2 - 2ab \cos\theta$$

$$V_s = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{r_1} - \frac{\alpha}{r_2} \right)$$

Require  $V_s = 0$  on surface

$$\Rightarrow \frac{1}{(a^2+r^2-2ar\cos\theta)^{1/2}} = \frac{\alpha}{(a^2+b^2-2ab\cos\theta)^{1/2}}$$

$$\alpha^2(a^2+r^2-2ar\cos\theta) = a^2+b^2-2ab\cos\theta$$

$$\Rightarrow \alpha^2(a^2+r^2) = a^2+b^2$$

$$\alpha^2 ar = ab \Rightarrow \alpha^2 = \frac{b}{r}$$

$$\Rightarrow b(a^2+r^2) = r(a^2+b^2)$$

$$ba^2 + br^2 - ra^2 - rb^2 = 0$$

$$a^2(b-r) + br(r-a) = 0$$

$$(a^2 - br)(b-r) = 0$$

$$b = a^2/r ; \alpha^2 = a^2/r$$

$\therefore E$  from image charge at distance  $(b-r)$  is

$$E = \frac{\alpha q}{\epsilon_0 4\pi (b-r)^2}$$

$$\therefore F = qE = \frac{\alpha q^2}{4\pi\epsilon_0 (b-r)^2} = \frac{a^2/r q^2}{4\pi\epsilon_0 (a^2/r - r)^2} = \frac{a^2 q^2}{4\pi\epsilon_0 (a^2 - r^2)^2}$$

BB — Plane wave in conductive medium

a)  $\nabla \times \underline{H} = \underline{J} + \underline{\dot{D}}$   
 $= \frac{1}{\mu_0} \underline{E} - i\omega \epsilon_r \epsilon_0 \underline{E} = -i\omega \epsilon_{\text{eff}} \epsilon_0 \underline{E}$

(if no conduction)  
 $\nabla \times \underline{H} = -i\omega \epsilon_r \epsilon_0 \underline{E}$

Hence  $\epsilon_{\text{eff}} = \epsilon_r (1 + \frac{i\omega}{\omega_0 \epsilon_r})$

$$\begin{aligned} k &= \frac{\omega}{v} = \frac{\omega}{\sqrt{\mu_0 \epsilon_0}} \\ &= \omega \sqrt{\mu_0 (\epsilon \epsilon_0 + \frac{1}{\omega_0^2})} \\ &\sim \omega \sqrt{\frac{\epsilon \mu_0}{k_0}} \quad \text{if } \epsilon \text{ large} \\ &= \sqrt{\frac{\epsilon_0 \mu_0 \omega}{2}} (1+i) \\ &= \frac{1+i}{8} : \delta = \sqrt{\frac{2}{\epsilon_0 \mu_0 \omega}} \end{aligned}$$

$$\begin{aligned} E &\stackrel{?}{=} E_0 e^{i(kx-\omega t)} \\ &= E_0 e^{i(x/8 - \omega t)} e^{-2x/8}. \end{aligned}$$

b) Power absorption

dispath =  $\underline{J} \cdot \underline{E} = \frac{G E^2}{2} \cos^2(\frac{x}{8} - \omega t) e^{-2x/8}$

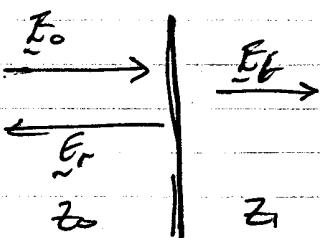
time average  $\langle \underline{J} \cdot \underline{E} \rangle = \frac{G E_0^2}{2} e^{-2x/8}$

Per unit area  $= \frac{G E_0^2}{2} \int_0^\infty e^{-2x/8} dx.$   
 $= \frac{G E_0^2}{2} \cdot \frac{8}{2} = G E_0^2 \delta / 4$

c) See over

B8 (ctd)

c) Power reflectivity:



$$\text{B.C.: } E_0 + E_r = E_t$$

$$H_0 - H_r = H_t \Rightarrow \frac{E_0}{Z_0} - \frac{E_r}{Z_0} = \frac{E_t}{Z_1}$$

$$\Rightarrow 2E_0 = E_t + E_r \left( \frac{Z_0}{Z_1} \right)$$

$$E_r = \frac{2E_0 Z_1}{Z_1 + Z_0} //$$

$$Z_1 = \sqrt{\frac{\mu}{\epsilon}}$$

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} = Z_0 \sqrt{\frac{\mu \epsilon_0}{\epsilon_0}}$$

Use  $E_t$  as  $E_0$  of part (b)

$$\text{Power absorbed/unit area} = \frac{G_s}{4} \cdot E_t^2 = \frac{G_s}{4} \cdot \left( \frac{E_t^2 Z_1^2}{Z_1 + Z_0} \right)^2$$

$$\text{if } G \text{ large} \rightarrow \approx \frac{E_t^2}{Z_0} \sqrt{\frac{2\mu \epsilon_0}{G}}$$

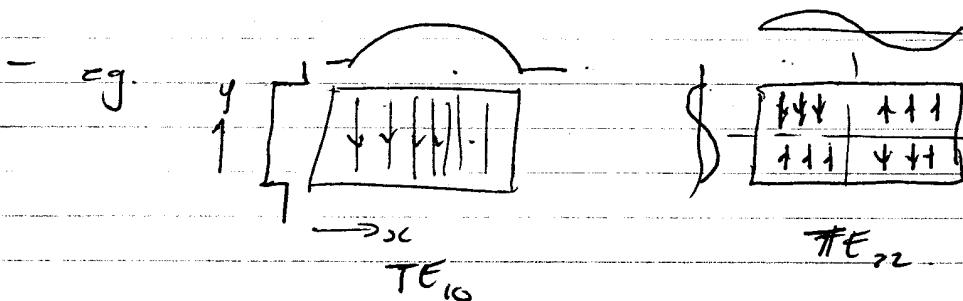
$$\text{In each flur} \propto \frac{E_t^2}{2Z_0}$$

$$\text{So fraction of power absorbed is } 2 \sqrt{\frac{2\mu \epsilon_0}{G}}$$

$$R = 1 - 2 \sqrt{\frac{2\mu \epsilon_0}{G}} //$$

## B9 — Waveguide

- a) - a w/g is a conductive tube - usually of rectangular or cylindrical cross-section.
- Wave obeys boundary conditions  
⇒ constraint on  $k_x, k_y$
- Different for TE ( $\Rightarrow E_z = 0$ )  
TM ( $\Rightarrow H_z = 0$ )



### b) Phase velocity

$$\omega^2 = c^2 k^2 = c^2 (k_x^2 + k_y^2 + k_z^2)$$

$$B.C \Rightarrow k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

$$\text{Hence } \omega^2 = k_z^2 c^2 + c^2 \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

$$= k_z^2 c^2 + \omega_{mn}^2 \quad *$$

$$v_p^2 = \omega^2 / k_z^2 = c^2 + \frac{\omega_{mn}^2}{k_z^2} \geq c^2 \quad //$$

c) Modes.  $16 \text{ GHz}, \lambda = \frac{3 \times 10^8}{16 \times 10^9} \approx 19 \text{ mm}$

w/g  $10 \times 20 \text{ mm}$

$$\omega_{mn}^2 = c^2 \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = \left( 15 \sqrt{\frac{m^2}{4} + n^2} \text{ GHz} \right)^2$$

$v_{c,10}$	=	7.5 GHz	TE	✓
$v_{c,01}$	=	15 GHz	TE	✓
$v_{c,20}$	=	15 GHz	TE	✓
$v_{c,11}$	=	18.75 GHz	TM	✗

(ftd.)

B9 (d)

d) Group velocity,  $v_g = \frac{\partial \omega}{\partial k_x}$ .

$$\text{From } * \quad \omega^2 = k_x^2 c^2 + \omega_m^2$$

$$2\omega \frac{\partial \omega}{\partial k_x} = 2k_x c^2$$

$$v_g = c^2 \frac{k_x}{\omega} = \frac{e^2}{m_p}$$

$$v_g(10) = 0.883c$$

$$v_g(0.1, 30) = 0.348c$$

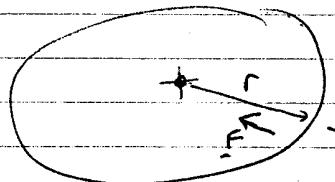
$$\Delta T = \frac{c}{v_g} \left( \frac{1}{0.348} - \frac{1}{0.883} \right)$$

$$= 8.8 \times 10^{-7} \text{ s.} //$$

1999 P1

Q

Q10



$$F = -Ar^{n-1}\hat{r}$$

1)  $\oint \mathbf{L} \cdot d\mathbf{r} = 0$  (conservation of angular momentum)

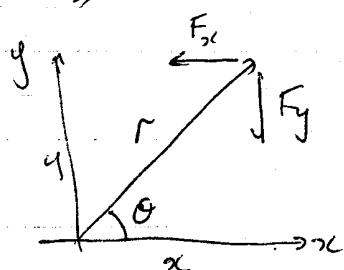
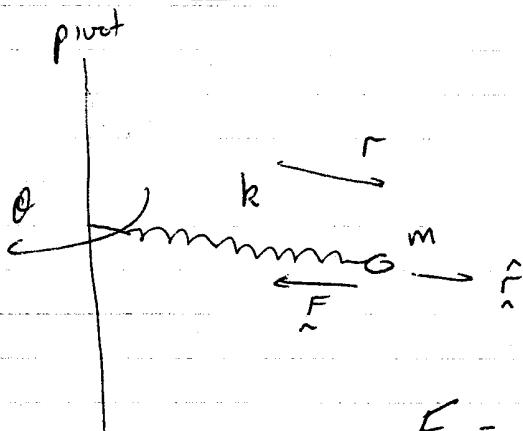
$$2) E = T + V \quad \therefore T = \frac{1}{2}mv^2 = \frac{1}{2}m(r^2 + v_\theta^2)$$

$$\begin{aligned} F &= -\nabla V \\ \Rightarrow V &= +Ar^{n-1}/(n-1) \quad (+\text{constant}) \end{aligned}$$

Here  $E = \frac{1}{2}mr^2 + \frac{1}{2}mv_\theta^2 + \frac{A}{n-1}r^{n-1}$   
 rework  $\frac{1}{2}mr^2 = E - \left(\frac{A}{n-1}r^{n-1} + \frac{1}{2}mv_\theta^2\right)$

But  $rv_\theta = \frac{J}{mr} \Rightarrow \sigma = mr^2v_\theta \quad \underbrace{U(r)}$

$$\begin{aligned} \text{so } \frac{1}{2}mr^2 &= E - \left(\frac{A}{n-1}r^{n-1} + \frac{J^2}{2mr^2}\right) \\ \boxed{\frac{1}{2}mr^2} &= E - U(r) \end{aligned}$$



$$\left. \begin{aligned} F_x &= -kx \\ F_y &= -ky \end{aligned} \right\} \text{sum : } \omega^2 = k/m$$

$$\text{i.e. } x(t) = a \cos(\omega t + \phi_x)$$

$$y(t) = b \cos(\omega t + \phi_y)$$

C10 contd

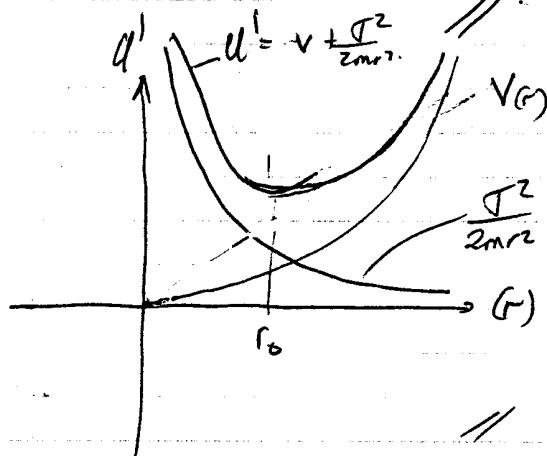
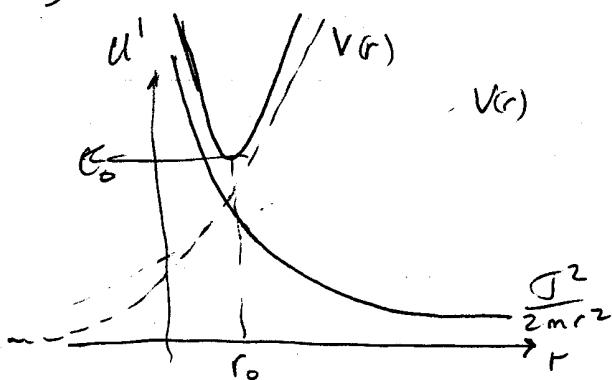
$$\text{if } \phi_x = 0, \\ \phi_y = \pi/2$$

$$\Rightarrow x(t) = a \cos \omega t \\ y(t) = b \sin \omega t \quad //.$$

Elliptical?

$$\frac{x^2}{a^2} = \cos^2 \omega t \\ \frac{y^2}{b^2} = \sin^2 \omega t$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \text{elliptical!}$$

 $U'(r)$ ?Function of  $\frac{J^2}{2mr^2}$ Modified force law,  $F = -k(r+d)\hat{r}$  [ie, more realistic!]Circular orbit? need minimum of  $U'(r)$ 

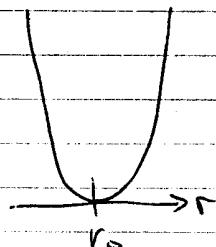
$$U'(r) = \frac{J^2}{2mr^2} + \frac{1}{2}k(r+d)^2 + \text{const.}$$

$$\frac{d}{dr}U'(r) = -\frac{J^2}{mr^3} + k(r+d)$$

$$r_0 : \boxed{\frac{J^2}{mr_0^3} = k(r_0+d)} : \frac{J^3(r_0+d)}{r_0^2} = \frac{J^2}{mr_0}$$

C10 (contd)

$r_0 \gg d$ ?



oscillates in potential well

$$U'(r) \sim U'(r_0) + (r-r_0)U''(r_0) + \frac{1}{2}(r-r_0)^2 U'''(r_0)$$

at  $r=r_0$ ,  $U''(r_0)=0$  (by defn)  
ignore constant

hence  $U'(r) \sim \frac{1}{2}(r-r_0)^2 U'''(r_0)$

if  $\phi = \frac{1}{2}kx^2$ ,  $\omega^2 = \frac{k}{m} \Rightarrow \omega^2 = \frac{1}{m}U'''(r_0)$

and  $U'''(r_0) = \frac{3k^2}{mr_0^3} + k$

use earlier result:  $\frac{\sigma^2}{mr_0^3} = k(C_0+d)$

$$\Rightarrow \omega^2 = 4 \frac{k}{m} + \frac{3kd}{mr_0}$$

$$\boxed{\omega^2 = 4 \frac{k}{m} \left(1 + \frac{3d}{r_0}\right)}$$

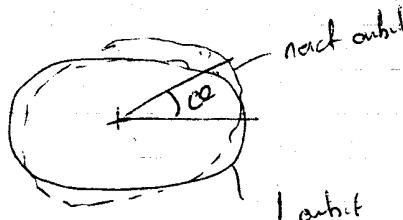
Orbital period?

for unperturbed orbit,  $\omega = \sqrt{\frac{k}{m}}$

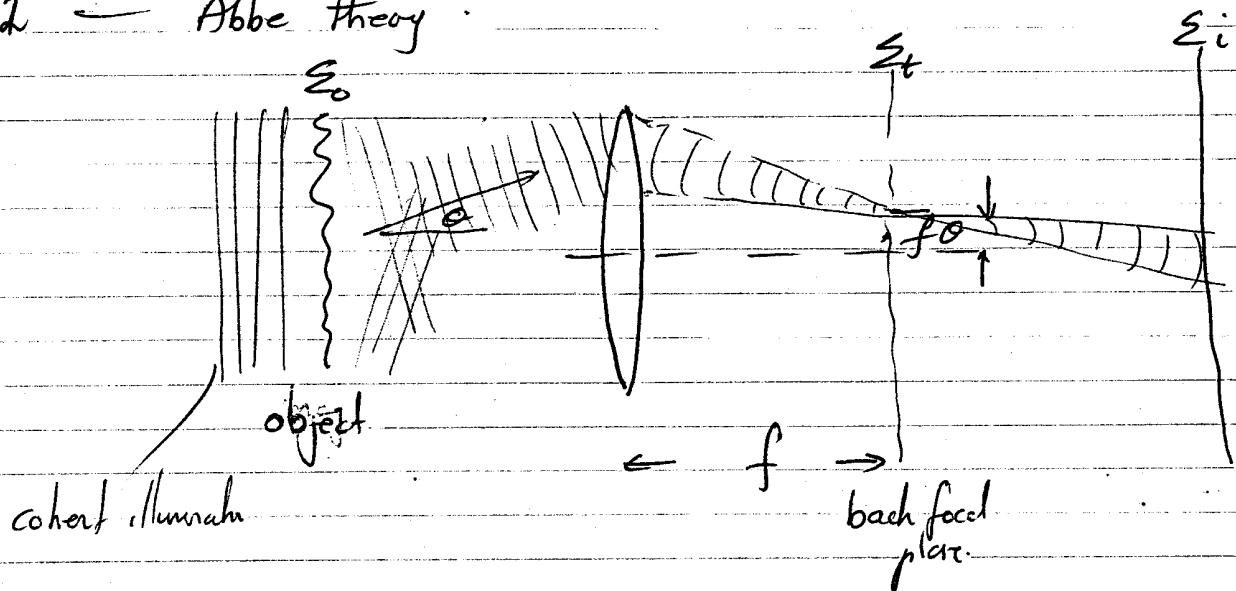
hence  $\omega \approx 2\pi \Rightarrow$  orbit precesses

Prec. speed  $= \frac{2\pi(\omega/2 - \omega)}{\omega}$  radian per orbit  
 $= 2\pi \left( \sqrt{1 + \frac{3d}{r_0}} - 1 \right)$

If  $r_0 \gg d$ ,  $\theta = \frac{3\pi d}{4r_0}$  radians

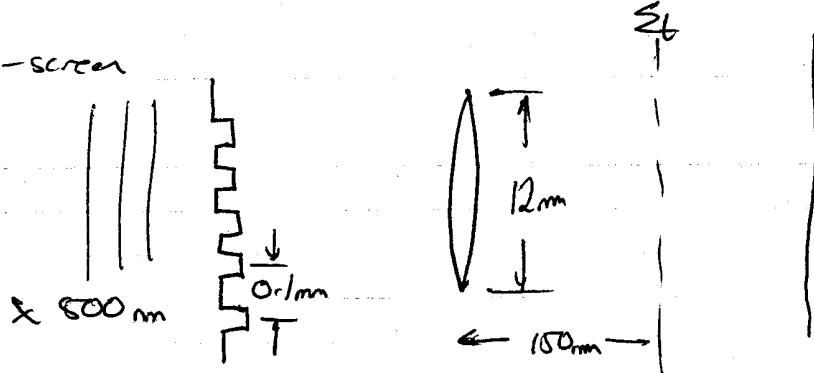


## 12 - Abbe theory.



- view obj as a general diffraction grating
  - plan wave at various angle nedd a lens  
— high order wave uses lens, ord are lost
  - each plan wave focused to a separate point in the back focal plane  
— plan wave expand again, ord overlap in image plane
- ⇒ Image arises from diffraction process.

## 2) Phase-screen



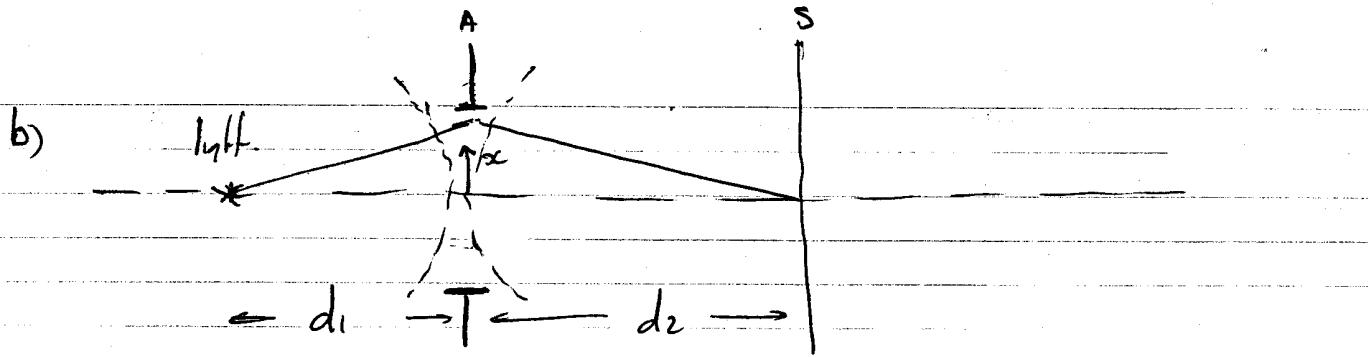
- Diffraction pattern at  $S_t$  :

$$\rightarrow \frac{\lambda F}{d}$$

$$= 0.75 \text{ mm}$$

- Spot size limited by quality of lens  $\approx \frac{1.2 \lambda F}{D_{lens}} = 0.0075 \text{ mm}$

- Darkfield screen: cut out central spot on back focal plane.  $D_{dark} > 0.0075 \text{ mm}$   
and  $D_{dark} < 1.5 \text{ mm}$ .



$$\text{Total path} \cong \frac{x^2}{2d_1} + \frac{x^2}{2d_2} = \frac{x^2}{2R_{\text{eff}}} : R_{\text{eff}}(d_1 + d_2)$$

$$\text{Radius of } n^{\text{th}} \frac{1}{2}\text{-period zero} = \frac{x^2}{2R_{\text{eff}}} = n\lambda/2 = \frac{n^2}{2R_{\text{eff}}}$$

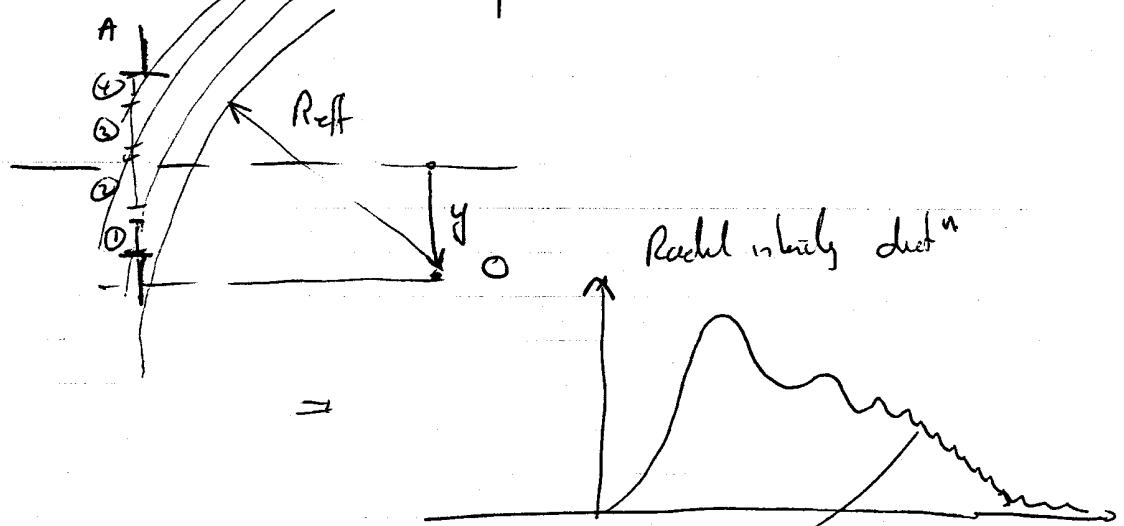
$$\Rightarrow g_n = \sqrt{n} \lambda R_{\text{eff}}$$

e)  $x_{\max} = 0.5 \text{ m}$   
 $d_1 = d_2 = 0.7 \text{ m} \Rightarrow R_f = 0.25 \text{ m}$   
 $x = 500 \text{ mm}$

$$P_{\text{max}} = \frac{x_{\text{max}}^2}{R_{\text{eff}}\lambda} = \frac{25 \times 10^{-8}}{0.25 \times 5 \times 10^{-7}} \\ = 2 \quad (\text{exactly})$$

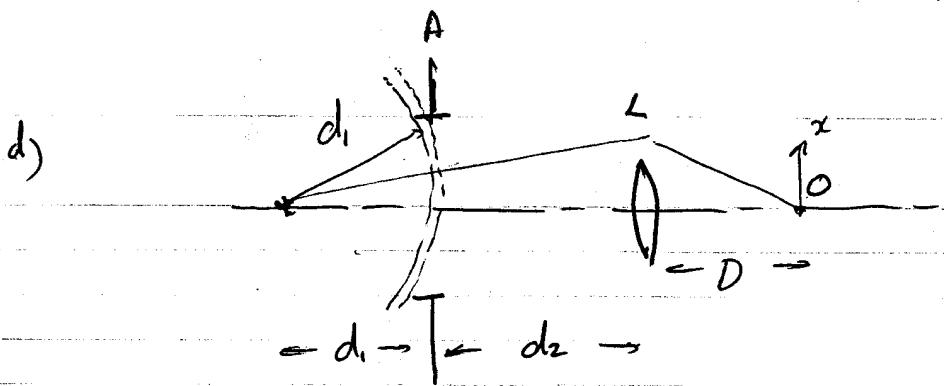
- So

  - 1) Exactly 2 zones  $\Rightarrow$  zero axial intensity (see diagram n(c))
  - 2) radially symmetric pattern (face symmetry of system)
  - 3) When moving off-axes at oblique point, we expose third zone or one side, and cover up 2<sup>nd</sup> zone on other



- x) angles increase in frequency, and decrease in amplitude.

5



d)

i) Geometrical image of grating:

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

$$\Rightarrow v = \frac{uf}{fu - f} = 0.75\text{m} \text{ (to right of lens)}$$

ii) Fraunhofer diffraction pattern: (see diagram above)

Choose distance "D" such that phase ~~over~~ over grating varies linearly or 0 more off-axes.

$\Rightarrow$  radius of curvature at A is same as due to source.

So Fraunhofer pattern will occur where the source is imaged.

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}; \frac{1}{u} = 1\text{ m}^{-1}$$

$$\frac{1}{v} = \frac{uf}{fu - f} = 0.429\text{ m} \text{ (to right of lens)}$$