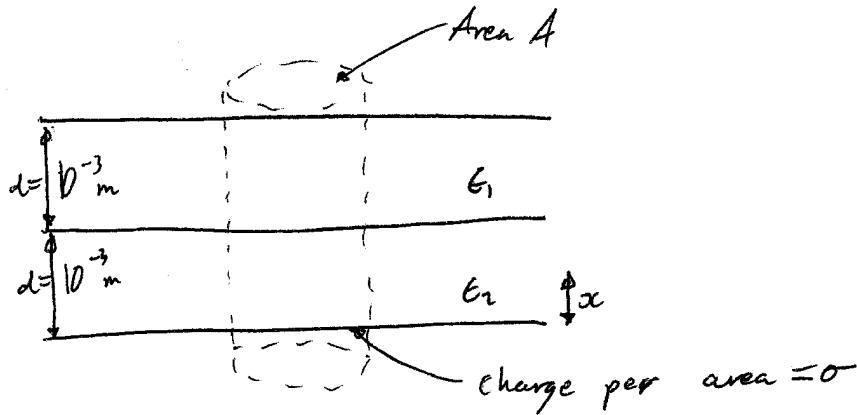


**2000**

*Paper 1*

2000 I

1)

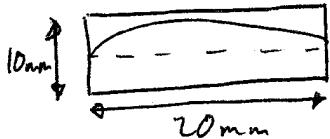


$$\text{Gauss : } DA = \sigma A$$

$$V = - \int_0^{10^{-3}} E_1 dx - \int_{10^{-3}}^{2 \times 10^{-3}} E_2 dx \\ = \frac{\sigma d}{\epsilon_0 \epsilon_1} + \frac{\sigma d}{\epsilon_0 \epsilon_2} = \frac{d (\epsilon_1 + \epsilon_2) \sigma}{\epsilon_0 \epsilon_1 \epsilon_2}$$

$$\frac{Q}{A} = \frac{CV}{A} = \sigma \Rightarrow \frac{C}{A} = \frac{\sigma}{V} = \frac{\epsilon_0 \epsilon_1 \epsilon_2}{d (\epsilon_1 + \epsilon_2)} = 16.6 \text{ nF m}^{-2}$$

2)

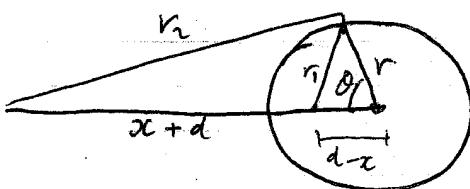
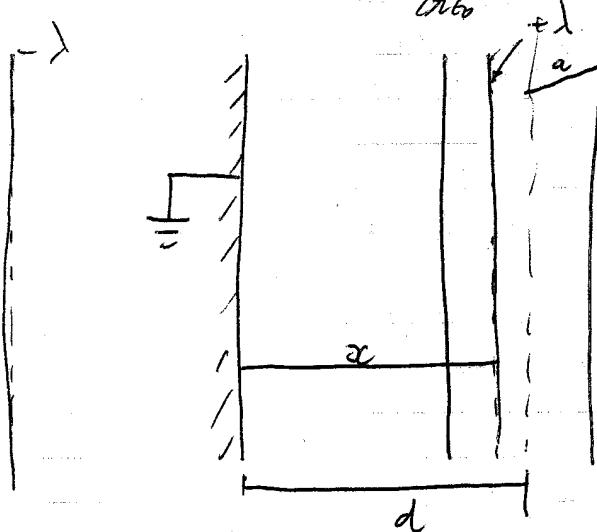


Minimum cutoff frequency for TE<sub>10</sub> mode  
 $\therefore \frac{\lambda_{\text{cut}}}{2} = 20 \text{ mm} \Rightarrow \lambda_{\text{cut}} = 4 \times 10^{-2} \text{ m}$   
 $\Rightarrow f_{\text{cut}} = 7.5 \text{ GHz}$

$$2000[6] \quad S.E.dA = \frac{L\lambda}{\epsilon_0}$$

$$2\pi r L E = \frac{L\lambda}{\epsilon_0}$$

$$V = \int E dr = \int \frac{2\pi r \epsilon_0}{\lambda} = -\frac{\lambda}{2\pi \epsilon_0} \ln r + \text{const}$$



$$r_1^2 = a^2 + (d-x)^2 - 2a(d-x) \cos\theta = A + B \cos\theta$$

$$r_2^2 = a^2 + (d+x)^2 - 2a(d+x) \cos\theta = C + D \cos\theta$$

$$V_s = -\frac{\lambda}{2\pi \epsilon_0} (\ln r_1 - \ln r_2) = \frac{\lambda}{2\pi \epsilon_0} \ln \left( \frac{C + D \cos\theta}{A + B \cos\theta} \right)$$

In order for this to have no  $\theta$  dependence

$$\Rightarrow \frac{D}{C} = \frac{B}{A} \Rightarrow \frac{2a(d-x)}{a^2 + (d-x)^2} = \frac{2a(d+x)}{a^2 + (d+x)^2}$$

$$(d-x)(a^2 + d^2 + x^2 + 2xd) = (d+x)(a^2 + d^2 + x^2 - 2xd)$$

$$2xd^2 - xd^2 - xd^2 - x^3 = -2xd^2 + xa^2 + xd^2 + x^3$$

$$\Rightarrow x^3 + xd^2 + a^2 x - 2xd^2 = 0$$

$$\Rightarrow x^2 = d^2 - a^2 \Rightarrow x = \sqrt{d^2 - a^2}$$

$$\therefore V_s = \frac{\lambda}{2\pi\epsilon_0} \ln \left( \frac{D}{B} \right)$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{\frac{D}{B}}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{\frac{(x+d)}{(d-x)}}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{\frac{(x+d)^2}{(d^2-x^2)}}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \sqrt{\frac{(x+d)}{a^2}}$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \left( \frac{d+x}{a} \right)$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln \left( \frac{d + \sqrt{d^2 - a^2}}{a} \right)$$

$$V = 10^4 \text{ V}$$

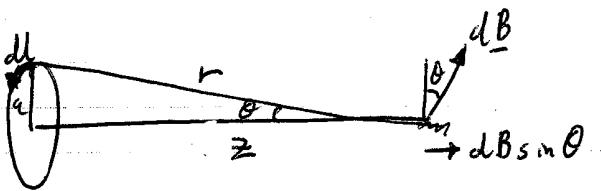
$$d = 10 \text{ m}$$

$$a = 0.02 \text{ m} \quad \Rightarrow \lambda = \frac{2\pi\epsilon_0 V_s}{\ln 1000}$$

$$= 8.05 \times 10^{-8} \text{ C m}^{-1}$$

$$F = qE = \frac{\lambda^2}{2\pi(2d)\epsilon_0} = 5.8 \times 10^{-6} \text{ N m}^{-1} \quad \text{Attractive.}$$

2000 I 7.



Components in all dims other than  $\parallel$  cancel.

$$dB_{\parallel} = dB \sin \theta = \frac{\mu_0 I}{4\pi} \frac{dl}{r^2} \sin \theta \quad \text{because } |dl \times r| = r dl \\ \text{and } \sin \theta = \frac{a}{r}$$

$$B_{\parallel} = \int dl B_{\parallel} = \frac{\mu_0 I a^2}{2 r^3}$$

$$\nabla \cdot \underline{B} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_{\rho}) + \frac{1}{\rho} \frac{\partial B_{\phi}}{\partial \phi} + \frac{\partial B_z}{\partial z} = 0 \quad \frac{\partial B_z}{\partial z} = 0$$

$$\therefore \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_{\rho}) = - \frac{\partial}{\partial z} \left( \frac{\mu_0 I a^2}{2 r^3} \right) \\ = + \frac{3z \mu_0 I a^2}{2 (z^2 + a^2)^{5/2}}$$

$$\therefore \frac{\partial}{\partial \rho} (\rho B_{\rho}) = + \frac{3z \mu_0 I a^2}{2 (z^2 + a^2)^{5/2}}$$

$$\rho B_{\rho} = \int \frac{+3z \mu_0 I a^2}{2 (z^2 + a^2)^{5/2}} = + \frac{3z \rho^2 \mu_0 I a^2}{4 (z^2 + a^2)^{3/2}}$$

$$\Rightarrow B_{\rho} = \frac{3 \mu_0 I a^2 z \rho}{4 (z^2 + a^2)^{5/2}}$$

$$\underline{F} = (\underline{m} \cdot \nabla) \underline{B} \quad \underline{m} = (0, 0, I \pi a^2) \quad \underline{m} \cdot \nabla = I \pi a^2 \frac{\partial}{\partial z}$$

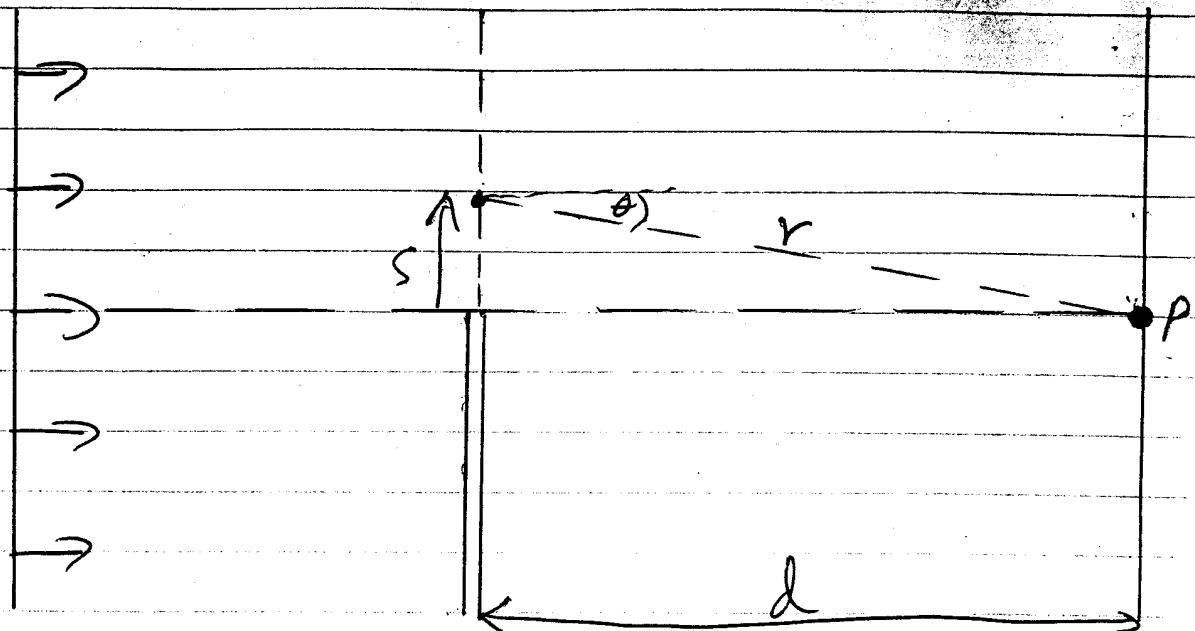
$$\therefore \underline{F} = I \pi a^2 \left( \frac{\partial}{\partial z} \left( \frac{3 \mu_0 I a^2 z \rho}{4 (z^2 + a^2)^{5/2}} \right), 0, \frac{-3z \mu_0 I a^2}{2 (z^2 + a^2)^{3/2}} \right) \\ = 0 \text{ at } \rho = 0$$

$$\Rightarrow |\underline{F}| = \frac{3 \pi \mu_0 I^2 a^4}{2 d^4} \quad \frac{z}{(z^2 + a^2)^{5/2}} \approx \frac{1}{d^4} \text{ for } d \gg a$$

Force is attractive.

(13)

Paper L



Assume that the light falling on the obstacle has a constant intensity, then the plane containing the obstacle can be divided into sources of wavelets of strength  $A$  per unit length, where  $A$  is an amplitude not intensity.

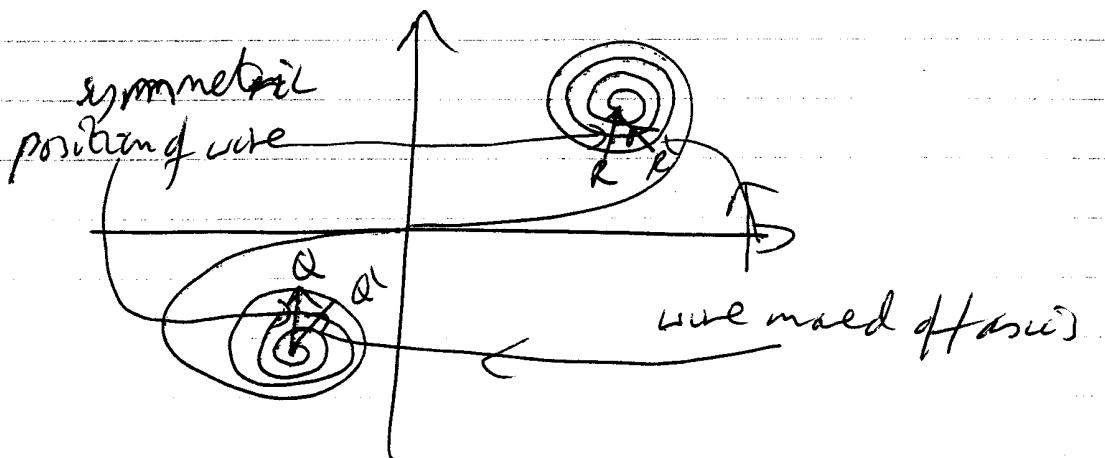
For a source at  $s$ , the amplitude at  $P$  is

$$d\psi_P = A ds \times k(\theta) \frac{e^{jkr}}{r} \times h(s)$$

where  $h$  is an aperture function,  $k(\theta)$  is the obliquity factor.

$$\begin{aligned} \text{Now } r &= \sqrt{s^2 + d^2} \\ &\approx d \left( 1 + \frac{s^2}{2d^2} \right) \quad \text{if } s \ll d \\ &= d + \frac{s^2}{2d}. \end{aligned}$$

For the wire, if fringes are observed inside the geometrical shadow, they arise because of interferences between the two contributions from either side of the wire :-



Initially when the wire is symmetric w.r.t. the screen the two contributions are in phase, but as we move off axis the two contributions go out of phase as above.

If the wire extends from  $-s$  to  $+s$

then the points  $Q$  &  $R$  correspond to  $-t$  &  $t$

if we move the ~~wire~~ wire by  $\Delta s$  in the positive direction

$Q'$  &  $R'$  correspond to  $\Delta s - s$ , &  $\Delta s + s$

i.e.  $\Delta t - t$  &  $\Delta t + t$

$$\begin{aligned} \text{The two corresponding phases are } \phi_+ &= \frac{\pi}{2} (1 + (t + \Delta t)^2) \\ &\approx \frac{\pi}{2} (1 + t^2(1 + 2\Delta t/t)) \end{aligned}$$

and  $\phi = \frac{\pi}{2} (1 + (\Delta t - \epsilon)^2)$

$$\approx \frac{\pi}{2} \left( 1 + t^2 \left( 1 - \frac{2\Delta t}{t} \right) \right)$$

assuming  $t \gg \Delta t$

so the phase difference is

$$\begin{aligned}\Delta\phi &= \frac{\pi}{2} \left( 1 + (t^2 + 2\Delta t) - 1 \right. \\ &\quad \left. - (t^2 - 2\Delta t) \right) \\ &= 2\pi \Delta t.\end{aligned}$$

For the fringes to go from one maximum to another as the point of observation changes (or the wave moves) the phase must change by  $2\pi$

Hence  $2\pi = 2\pi \Delta t$

so  $\Delta t = \frac{1}{f}$

Now  $t = s \sqrt{\frac{2}{\lambda d}}$

$$\Rightarrow \Delta t = Ds \sqrt{\frac{2}{\lambda d}}$$

We know that the fringes have a spacing of

$$0.05\text{mm} = 5 \cdot 10^{-5}\text{m}$$

Now  $\frac{l}{t} = Ds \sqrt{\frac{2}{\lambda d}}$

Substitute for  $t$

$$\Rightarrow \frac{l}{s \sqrt{\frac{\lambda d}{s}}} = Ds \sqrt{\frac{2}{\lambda d}}$$

$$\Rightarrow s = \frac{l}{Ds} \frac{\lambda d}{2}$$

$$= \frac{l}{5 \times 10^3} \frac{585 \times 10^{-9} \times 0.2}{2}$$

$$= 1.09 \text{ mm.}$$

Hence the wire diameter is  $2s = 2.18 \text{ mm.}$

If  $s$  is sufficiently smaller than  $d$  we can ignore the term in  $s^2/d^2$  for the denominator  $r$ , but when  $r$  appears in the exponential  $\exp ikr = \exp ikd \exp iks^2 \frac{1}{2d}$

we cannot since  $\exp ikd$  is constant as  $s$  varies whilst  $\exp iks^2 \frac{1}{2d}$  isn't.

Hence Assuming the angles are small so that  $k(s) \approx 1$  since  $\theta \sim s/d$  we have for the total amplitude at  $P$

$$\Psi_P = \frac{A}{d} \int_{-\infty}^{\infty} h(s) \exp ikd \exp iks^2 \frac{1}{2d} ds$$

$$\boxed{\Psi_P = \frac{A \exp i2\pi d}{d} \int_{-\infty}^{\infty} h(s) \exp i\frac{\pi s^2}{\lambda d} ds} *$$

\* the contribution from the elementary strips has a relative phase factor  $\exp i\frac{\pi s^2}{\lambda d}$

In  $\textcircled{*}$  the leading term is constant + the diffraction pattern is determined by the integral

For an edge or otherwise totally opaque obstacle, the diffraction pattern is determined by setting

$$h(s) = 0 \quad s_1 < s < s_2 \quad \text{where the edge is}$$

$$= 1 \quad s < s_1 \text{ or } s > s_2 \quad \text{where the obstacle isn't}$$

Let  $t = s\sqrt{\frac{2}{\lambda d}}$  then the integral becomes

$$\Psi_p = \text{const} \times \int_{t(s_1)}^{t(s_2)} \exp \frac{i \pi t^2}{2} dt \propto \int_{t_1}^{t_2} \exp \frac{i \pi t^2}{2} dt$$

$$\propto \int_{t_1}^{t_2} \exp \frac{i \pi t^2}{2} dt = \int_{t_1}^{t_2} \cos \frac{i \pi t^2}{2} dt + i \int_{t_1}^{t_2} \sin \frac{i \pi t^2}{2} dt$$

$$= \cancel{C(b)} + i S(b)$$

$$= \int_0^{t_2} \cos \frac{i \pi t^2}{2} dt + \int_{t_1}^0 \cos \frac{i \pi t^2}{2} dt$$

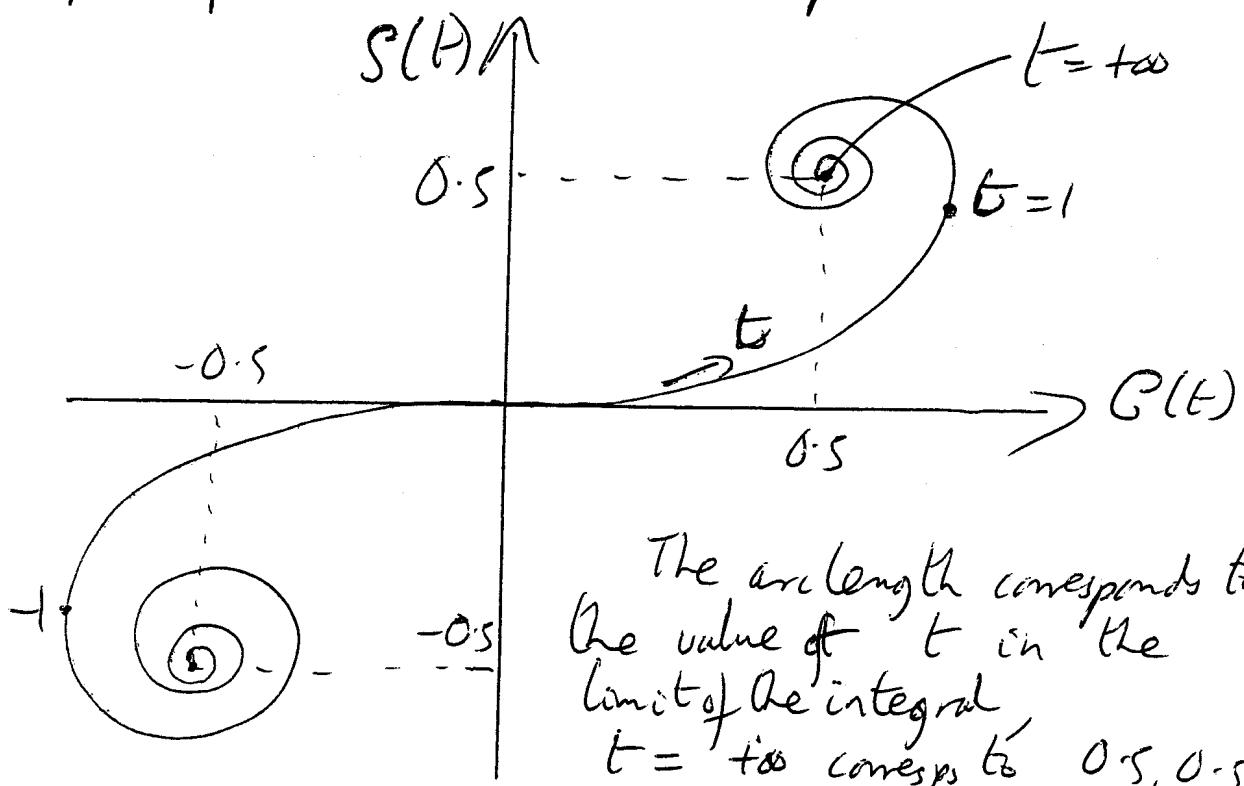
$$+ i \left( \int_0^{t_2} \sin \frac{i \pi t^2}{2} dt + \int_{t_1}^0 \sin \frac{i \pi t^2}{2} dt \right)$$

$$= C(t_2) + i S(t_2) - (C(t_1) + i S(t_1))$$

Where  $S(t) = \int_0^t \sin \frac{\pi t^2}{2} dt$

$$C(t) = \int_0^t \cos \frac{\pi t^2}{2} dt$$

The locus of the point  $C(t) + i S(t)$  in the complex plane is the Cornu spiral



(Not part of answer:  $dS(t) = \sin \frac{\pi t^2}{2} dt$

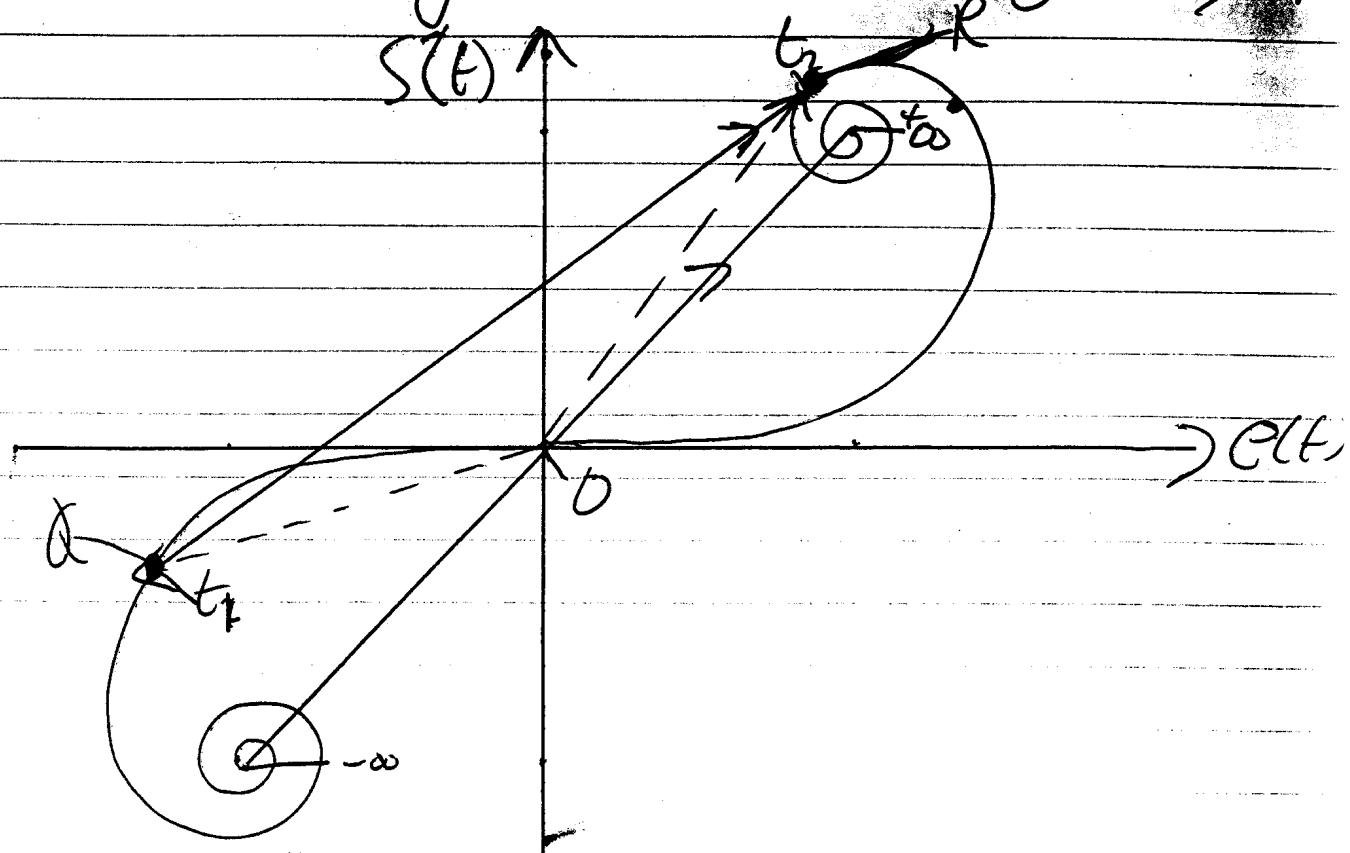
$$dC(t) = \cos \frac{\pi t^2}{2} dt$$

$$\text{arc length } (dl)^2 = (dS(t))^2 + (dC(t))^2$$

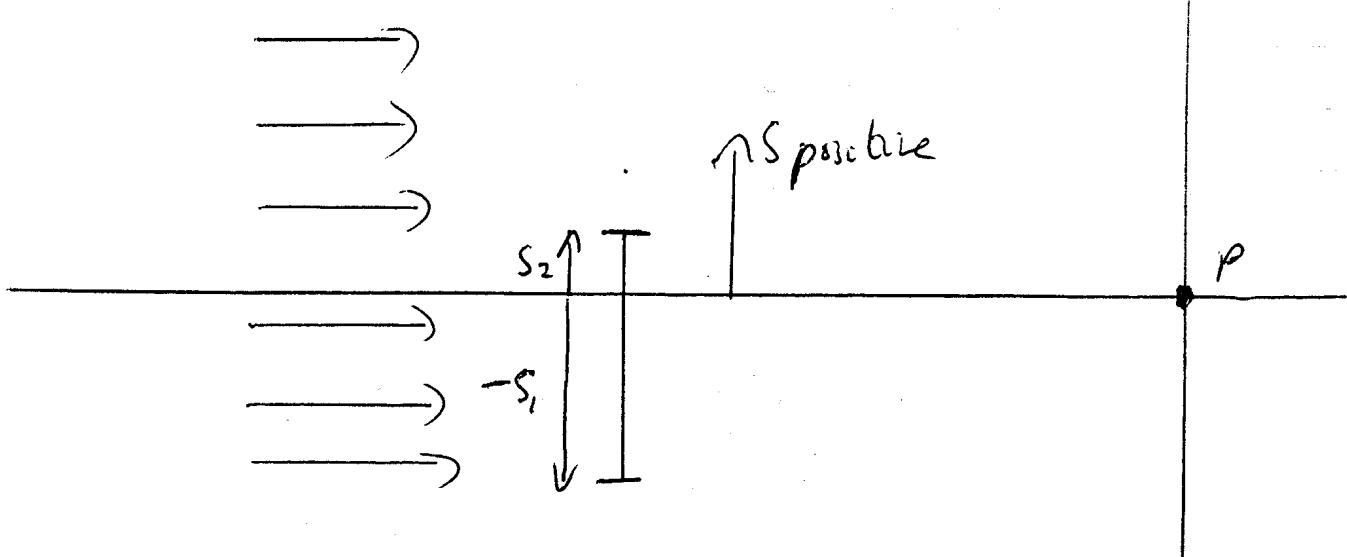
$$= \left( \sin^2 \frac{\pi t^2}{2} + \cos^2 \frac{\pi t^2}{2} \right) (dt)^2 = (dt)^2$$

hence arc length corresponds to value of  $t$  since  $dl = dt$

In order to find the ~~amplitude~~ at  $P_1$ ,  $P_2$



Use the values of  $s$  corresponding to the edges of the obstacle  $s_1 \times s_2$  as in the sketch below



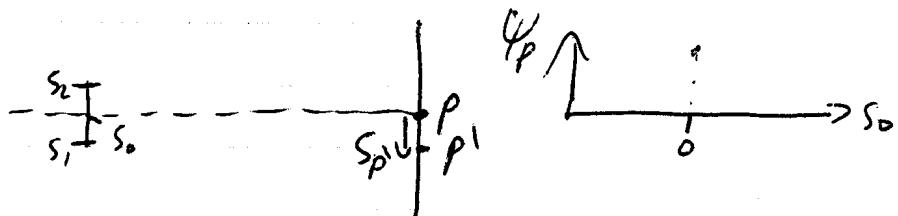
and obtain  $t_2 \times t_1$ , plot these on the corn spiral & draw the vector joining the two points which equals the difference between vectors joining OR & OQ in the figure at the top.

This is then normalized to the unobstructed amplitude to which is obtained by finding the length of the vector from  $t = -\infty$  to  $t = +\infty$ .

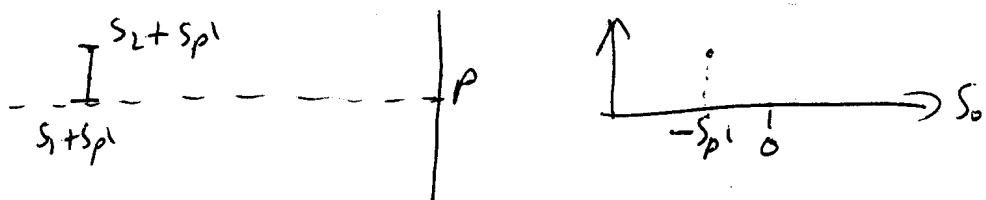
The normalized intensity is then  $\left(\frac{\psi_p}{\psi_{\infty}}\right)^2$ .

To obtain the whole pattern:

Begin with the obstacle at a point of high symmetry, e.g.  $|S_2| = |S_1| \quad S_0 = 0$

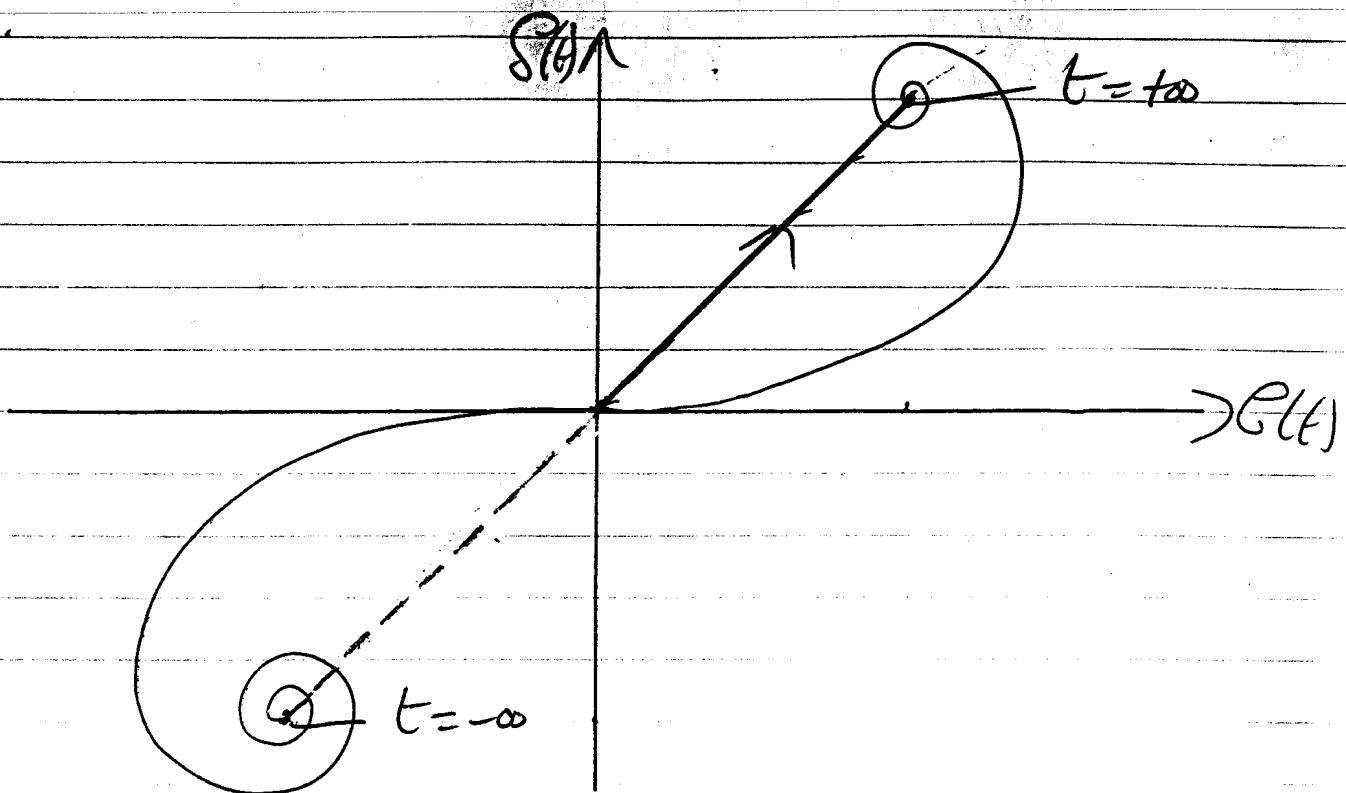


Then use the approximation that since rays are almost parallel amplitude at point  $P'$  can be found by moving the obstacle up by a distance  $s_{p'}$



Hence for each value of  $s_{p'}$ , a vector is plotted on the Cornu spiral corresponding to new values of  $t_2 \times t_1$  obtained from the new values of  $S_2 \propto S_1$ .

①.



For the edge in the position shown on the exam paper.

$$s_1 = 0 \quad s_2 = +\infty$$

$$\Rightarrow \text{since } t = \sqrt{\frac{2s}{\lambda d}}$$

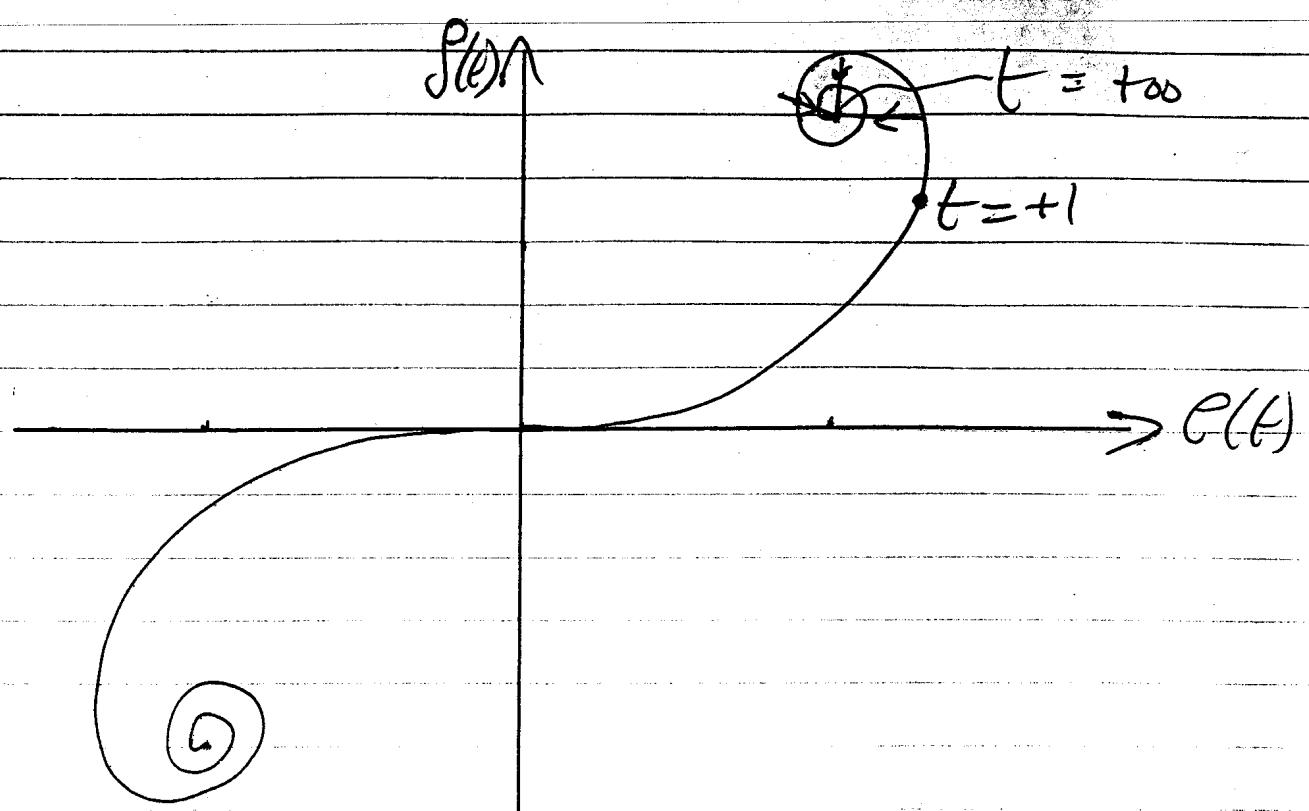
$$t_1 = 0 \quad t_2 = +\infty$$

Hence the amplitude is given by the vector from  $t = 0$  to  $t = \infty$  as above

This is half the length of the vector from  $t = -\infty$  to  $t = +\infty$  hence the amplitude at P is half

the unobstructed amplitude.

②



Deep inside the geometrical shadow  $t_1 > 1$  at  $t_2 = \infty$  this puts the vector deep inside the spiral centred on  $(0.5, 0.5)$ . As  $t$  increases, since the spiral is tightly wound, the length of the vector does not change much but its direction rotates as  $t$  increases. So the phase oscillates with position on the screen deep inside the shadow.

② continued. deep inside the geometrical

shadow, the

coma spiral is

fairly circular, so

The phase angle of

The vector is

$\frac{\pi}{2}$  + the angle that

The tangent makes with the radius of the circle.

The slope of the tangent is

$$\frac{ds(t)}{dc(t)} = \frac{\sin \pi t^2/2}{\cos \pi t^2/2} = \tan \frac{\pi t^2}{2}$$

Hence the phase angle of the vector is

$$\phi = \frac{\pi}{2} + \frac{\pi}{2} t^2 = \frac{\pi}{2} (1+t^2)$$

