

**2000**

*Paper 2*

# Advanced physics paper 2 question.

①. Intensity of a sound wave;

This is the mean power transmitted by the wave given by  $\frac{1}{2} \text{Re} [F \dot{u}^*]$

$F = \text{force}$       $u = \text{transverse velocity}$ .

But  $F = Z u$ , so

$$\begin{aligned} \text{the intensity is } & \frac{1}{2} \text{Re} [Z u \dot{u}^*] \\ & = \frac{1}{2} Z \dot{u}^2 \end{aligned}$$

$$\text{Now } u = \frac{\partial y}{\partial t} = i \omega A_0 \exp(i \omega t - kx)$$

$$\text{mean power} = \frac{1}{2} Z \omega^2 A_0^2$$

$$\text{for sound in a gas } Z = \rho c \quad c = \sqrt{\frac{\delta p}{\rho}}$$

$$\text{so } Z = \sqrt{\delta p \rho}$$

$$pV = nRT = NkT \quad n = \text{number of moles}$$

$$\text{so } p = \frac{\rho}{m} kT \Rightarrow \rho = \frac{mp}{kT}$$

$$p = 10^5 \text{ Pa}$$

$m = \text{mass of Nitrogen}$

or  $0.8 \text{ Mass N} + 0.2 \text{ mass O}$

$$r = ?$$

Or use  $c$  for air  $= 340 \text{ m s}^{-1}$

$$\text{then } \tau = \frac{c m p}{k T}$$

$$m \approx 0.8 \times 14 \times 1.66 \times 10^{-27} \\ + 0.2 \times 16 \times 1.66 \times 10^{-27} \\ \approx 2.34 \times 10^{-26} \text{ kg}$$

~~$$\Rightarrow \tau = \frac{340 \times 2.34 \times 10^{-26} \times 10^5}{8.1 \times 10^{-19}}$$~~

~~$$\omega = 2\pi f \quad f = 500 \text{ Hz} \\ \Rightarrow \omega = 3141 \text{ s}^{-1}$$~~

$$\tau = \frac{340 \times 2.34 \times 10^{-26} \times 10^5}{1.38 \times 10^{-23} \times 300} \\ = \underline{146.3 \text{ N s m}^{-1}}$$

$$\underline{\omega = 3141 \text{ s}^{-1}}$$

The displacement amplitude is:-

$$A_0 = \sqrt{\frac{2 \text{ Intensity}}{Z \omega^2}}$$

$$= \sqrt{\frac{2 \times 10^{-12}}{146.3 \times 3141}}$$

$$= 3.2 \text{ nm}$$

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(2)

## Asymptotic + Resonance

$$\begin{array}{l} \text{expika} \rightarrow \\ \text{resp-ika} \leftarrow \end{array}$$

$$\text{expika} \rightarrow$$

The reflection coefficient is proportional to  
 $r \propto (1 - \exp i 2k'a)$

$r$  hence is zero when

$$\exp i 2k'a = 1$$

$$\Rightarrow 2k'a = 2n\pi$$

$$\Rightarrow k' = \frac{n\pi}{a}$$

But  $E = \frac{\hbar^2 k'^2}{2m} + V_0$

$$\Rightarrow E - V_0 = \frac{\hbar^2}{2m} \left( \frac{n\pi}{a} \right)^2 \quad \text{for } r=0$$

$$\Rightarrow \text{for } n=1$$

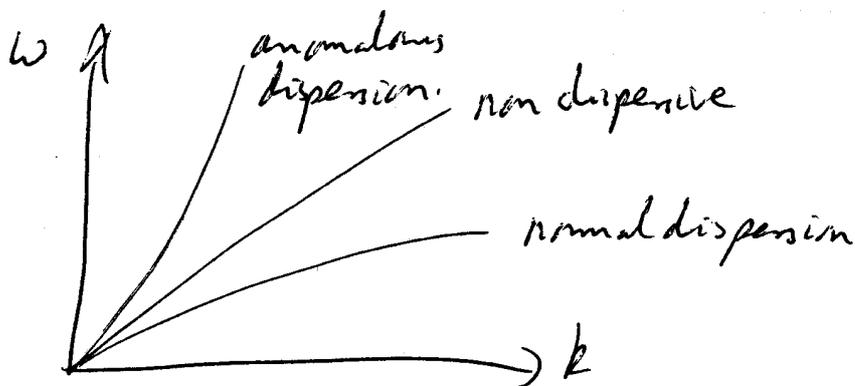
$$\begin{aligned} a &= \sqrt{\frac{\hbar^2 \pi^2}{2m(E - V_0)}} = \sqrt{\frac{\hbar^2 \pi^2}{2m_e(1.5e-)}} \\ &= 5\text{\AA} \end{aligned}$$

(6). Dispersive Wave. Such a wave has a non linear relationship between frequency  $\omega$  wavenumber. ~~the~~ The dispersion of light in a vacuum is of the form

$$\omega = c_0 k \quad \nu = 2\pi f$$

$$\propto k = \frac{2\pi}{\lambda}$$

However when light propagates through a medium such as glass the relation between  $\omega$  &  $k$  becomes non-linear



Then the velocity  $c = \frac{\omega}{k}$  becomes a function of  $k$ . This velocity is the phase velocity as it is the velocity with which a point of constant phase in a wave of fixed frequency travels.

For such a wave 
$$\psi = \psi_0 \exp i(kx - \omega t)$$

$$= \psi_0 \exp ik(x - ct)$$

So a point of constant  $x - ct$  moves with velocity  $c$  to the right.

2000 P.2. Q6

If a disturbance consists of a superposition of waves with a range of frequencies, then if they are centred around a particular frequency  $\omega_0$  the group of waves will travel with a velocity

$$v_g = \left. \frac{\partial \omega}{\partial k} \right|_{\omega_0} = \text{group velocity}$$

For a non dispersive wave, this equals the phase velocity  $\frac{\omega}{k}$ , but may be higher or lower than the phase velocity for a dispersive medium.

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$$\frac{\partial^2 \psi}{\partial x^2} - \alpha \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

$$\text{let } \psi = \exp i(\omega t - kx)$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial}{\partial x} (-ik\psi) = -ik \times ik\psi \\ &= -k^2 \psi \end{aligned}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = +k^2 \psi$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} (i\omega\psi) = -\omega^2 \psi$$

2000 P2. Q6

Hence the dispersion relation is

$$-k^2 \psi - \alpha k^4 \psi = \frac{1}{c^2} (-\omega^2 \psi)$$

$$k^2 + \alpha k^4 = \frac{\omega^2}{c^2} \Rightarrow \omega^2 = c^2 k^2 (1 + \alpha k^2)$$

$$\omega = ck(1 + \alpha k^2)^{1/2}$$

$$v_g = \frac{d\omega}{dk} = \frac{ck \times (2\alpha k)}{2(1 + \alpha k^2)^{1/2}} + c(1 + \alpha k^2)^{1/2}$$

$$= \frac{2\alpha ck^2}{2(1 + \alpha k^2)^{1/2}} + c(1 + \alpha k^2)^{1/2}$$

$$= \frac{\alpha ck^2}{(1 + \alpha k^2)^{1/2}} + c(1 + \alpha k^2)^{1/2}$$

for small  $\alpha$

$$v_g = \alpha ck^2 \left(1 - \frac{\alpha k^2}{2}\right) + c \left(1 + \frac{\alpha k^2}{2}\right)$$

$$= c \left(1 + \frac{3}{2} \alpha k^2 - \frac{\alpha^2 k^4}{2}\right)$$

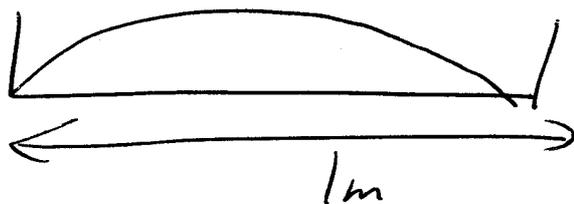
$$\approx c \left(1 + \frac{3}{2} \alpha k^2\right)$$

The <sup>2000 P2 Q6</sup> dispersion relation is

$$\omega = ck(1 + \alpha k^2)^{1/2}$$

we can use  $\omega^2 = c^2 k^2 (1 + \alpha k^2)$

For the fundamental



$$\lambda = 2l_m \text{ so } k = \frac{2\pi}{\lambda} = \pi.$$

$$\omega_0^2 = c_0^2 \pi^2 (1 + \alpha \pi^2) \quad (1)$$

For the first harmonic.  $\omega_1 = 2\omega_0$

$$\lambda = l_m \Rightarrow k = 2\pi.$$

$$4\omega_0^2 = c_1^2 4\pi^2 (1 + 4\alpha \pi^2)$$

$$\Rightarrow \omega_0^2 = \pi^2 c_1^2 (1 + 4\alpha \pi^2) \quad (2)$$

dividing  $1 = \left(\frac{c_1}{c_0}\right)^2 \frac{(1 + 4\pi^2 \alpha)}{1 + \pi^2 \alpha}$

$$1 + \alpha \pi^2 = \left(\frac{c_1}{c_0}\right)^2 (1 + 4\alpha \pi^2)$$

$$1 - \left(\frac{c_1}{c_0}\right)^2 = \alpha \pi^2 \left(4\left(\frac{c_1}{c_0}\right)^2 - 1\right)$$

$$\text{So } \alpha = \frac{1}{\pi^2} \frac{1 - (c_1/c_0)^2}{4(c_1/c_0)^2 - 1}$$

Now to keep  $\omega_1 = 2\omega_0$

$$\text{We have } T_1 = \frac{98.5}{100} T_0$$

$$\Rightarrow \left(\frac{c_1}{c_0}\right)^2 = \frac{98.5}{100}$$

$$\text{So } \alpha = \frac{1}{\pi^2} \frac{0.015}{2.94}$$

$$\alpha = 5.16 \times 10^{-4}$$

Could actually use approximation

$$\omega \approx ck \left(1 + \frac{\alpha k^2}{2}\right) \text{ since } \alpha \text{ is so small.}$$

Need to find  $k$  for  $f = \frac{\omega}{2\pi} = 50 \text{ Hz} \approx 25 \text{ Hz}$

Assume  $\omega = ck$

$$\text{Then } 50 \text{ Hz} \Rightarrow 314 \text{ s}^{-1}$$

$$\Rightarrow \text{since } c = 100 \text{ m s}^{-1}$$

$$k \approx 3.14 \text{ m}^{-1}$$

Insert into dispersion relation.

Correction to  $\omega$  would be 0.25%

So assume good enough.

200 p2 Q6

for  $25 \text{ Hz}$   $\omega = 157 \text{ s}^{-1}$

$$\Rightarrow k \approx 1.57 \text{ m}^{-1}$$

So the difference in group velocities for the two frequencies is

$$\Delta v_g = \frac{3c\alpha}{2} \left( (3.14)^2 - (1.57)^2 \right)$$

$$= \frac{3}{2} \times 100 \times 5.16 \times 10^{-8} \times 7.4$$

$$= 0.57 \text{ m s}^{-1}$$

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8.

## Boundary conditions for a wavefn.

$\Psi$  Since  $|\Psi|^2$  represents the probability density of finding the particle

$\Psi$  must be finite everywhere

$\Psi$  must be single valued at a point because otherwise  $\hat{p}\Psi = \frac{\hbar}{i} \frac{\partial \Psi}{\partial x}$  would produce a singular unphysical result.

here  $\Psi$  must be continuous if  $\Psi$  is single valued and finite

$\frac{d\Psi}{dx}$  Since  $\Psi$  is a solution of Schrödinger's equation

$$\frac{d^2\Psi}{dx^2} = -\frac{2m}{\hbar^2} (E - V(x)) \Psi(x)$$

$\times$   $\Psi$  is finite, if  $V(x)$  is finite  $\times E$  is finite

$\frac{d^2\Psi}{dx^2}$  must also be finite

$\times$  if  $\frac{d^2\Psi}{dx^2}$  is finite

$\frac{d\Psi}{dx}$  must be continuous

otherwise  $\frac{d^2\Psi}{dx^2}$  would be discontinuous

If  $V(x)$  is ~~discontinuous~~ not finite

then for solutions of the form

$\psi \propto \exp ikx$

$$E = \frac{\hbar^2 k^2}{2m} + V$$

for all finite  $E$ ,  $k^2 = \frac{2m(E-V)}{\hbar^2} < 0$

$\psi$  has the form  $\psi \propto \exp -Kx$

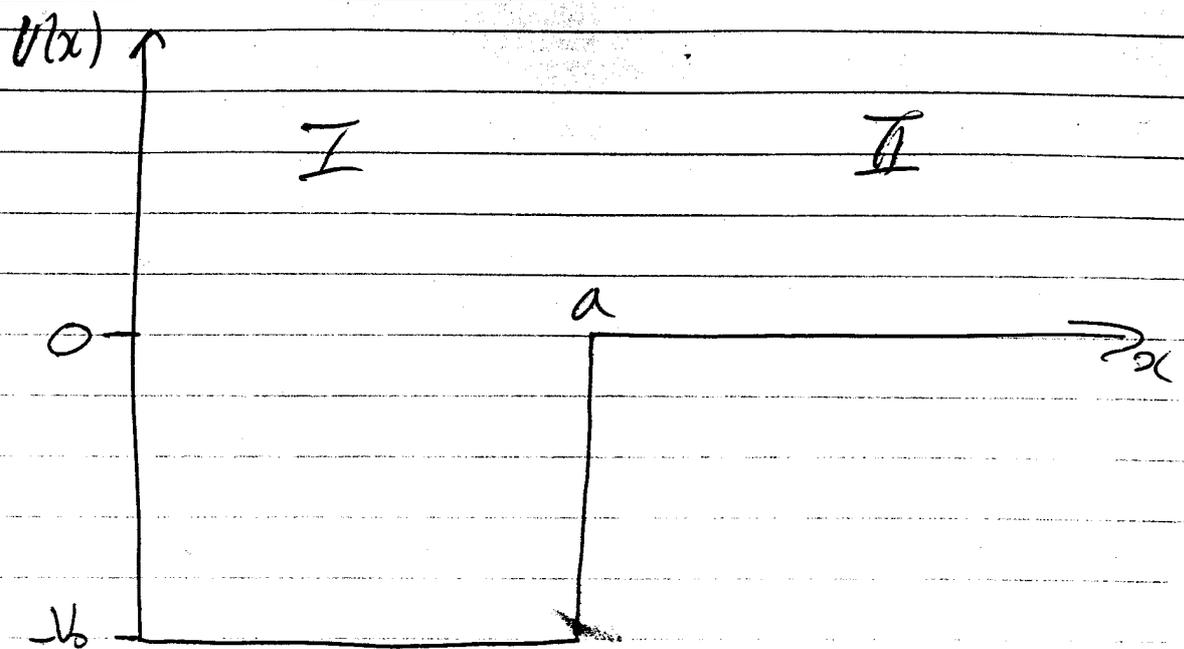
where  $K = \sqrt{\frac{2m(V-E)}{\hbar^2}}$

as  $V \rightarrow \infty$   $K \rightarrow \infty$

$\psi$  hence in a region where  $V$  is not finite

$\psi$  must equal zero, but is still continuous with the finite  $V$  region

So  $\frac{d\psi}{dx}$  is not continuous.



in I we have

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + (-V_0)\psi = E\psi$$

We can use solutions of the form

$$\psi = A \sin kx + B \cos kx$$

But since at  $x=0$   $\psi=0$

$$B=0, \text{ so } \psi = A \sin kx$$

$$\text{with } E = \frac{\hbar^2 k^2}{2m} - V_0$$

$$\times \frac{d\psi}{dx} = kA \cos kx$$

in II for Bound state solutions we

must have  $\psi \rightarrow 0$  at large  $x$

The Schrödinger equation is

$$E\psi = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2}$$

we try solutions of the form.

$$\psi = C \exp(-qx) \quad \text{which decays to zero for large } x$$

$$\text{so } \frac{d\psi}{dx} = -Cq \exp(-qx)$$

$$\text{hence } E = -\frac{\hbar^2 q^2}{2m}$$

At  $x = a$

$$\psi(a) \Rightarrow A \sin ka = C \exp(-qa)$$

$$\frac{d\psi}{dx}(a) \Rightarrow kA \cos ka = -Cq \exp(-qa)$$

~~by~~ dividing gives :-

$$k \cot ka = -q \quad \text{as required}$$

In order to determine the bound state energies we have two conditions

$$E = \frac{\hbar^2 k^2}{2m} - V_0 \quad \text{in I}$$

$$\propto E = -\frac{\hbar^2 q^2}{2m} \quad \text{in II}$$

$$\text{so} \quad \frac{\hbar^2}{2m} (k^2 + q^2) = V_0$$

which we can rewrite as

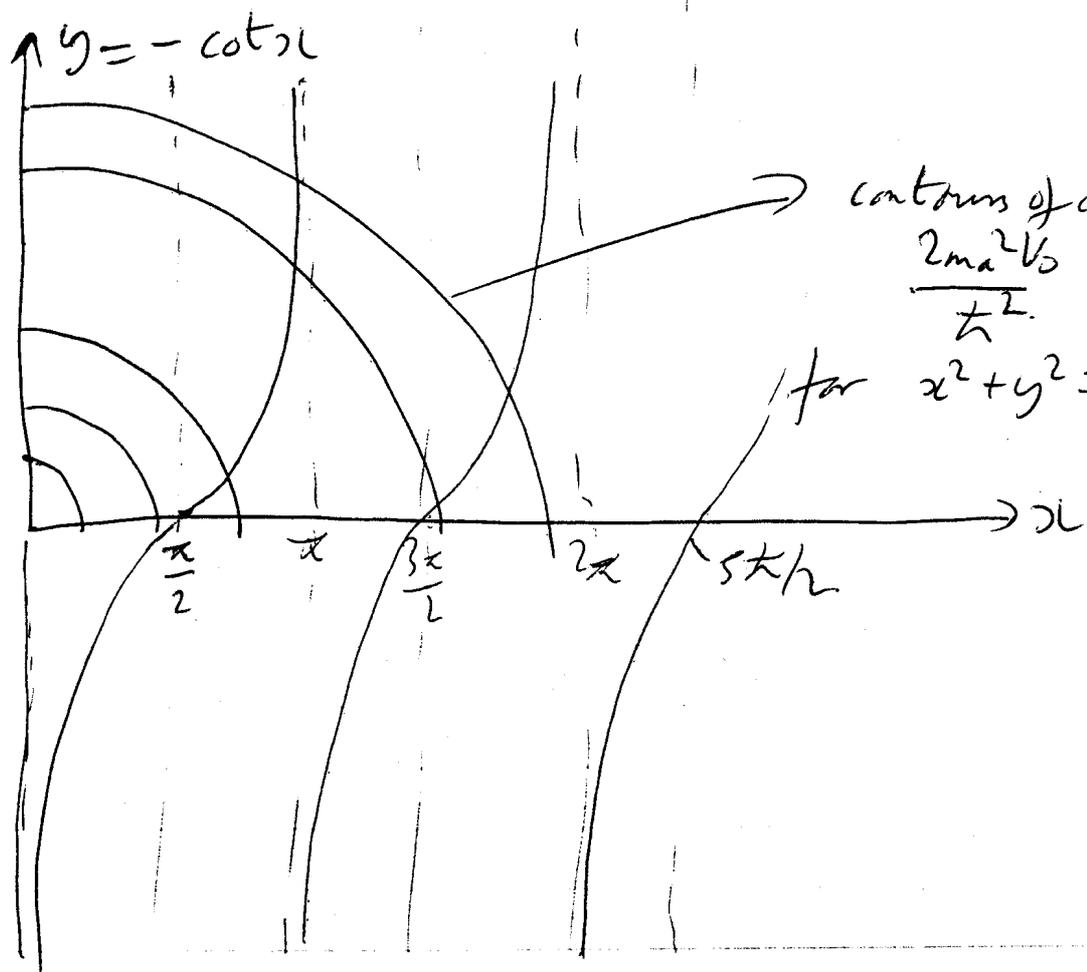
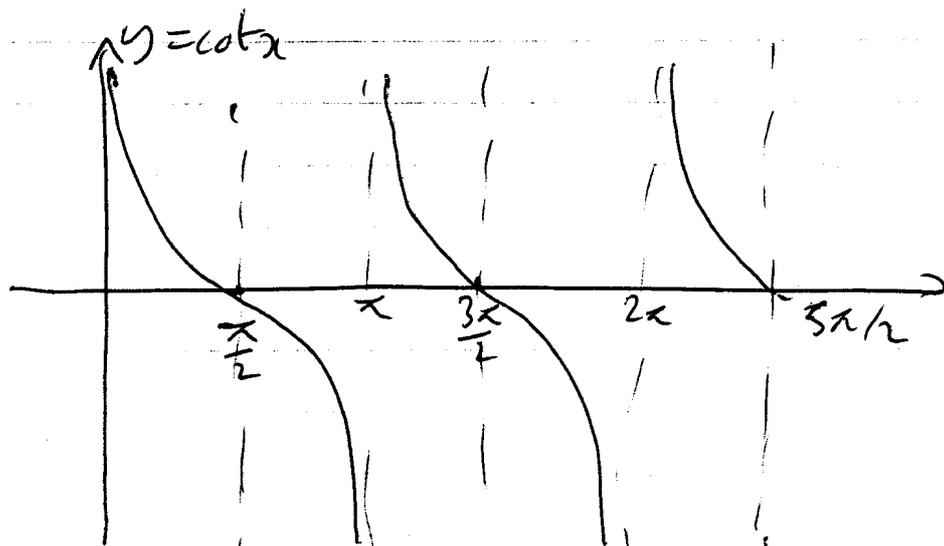
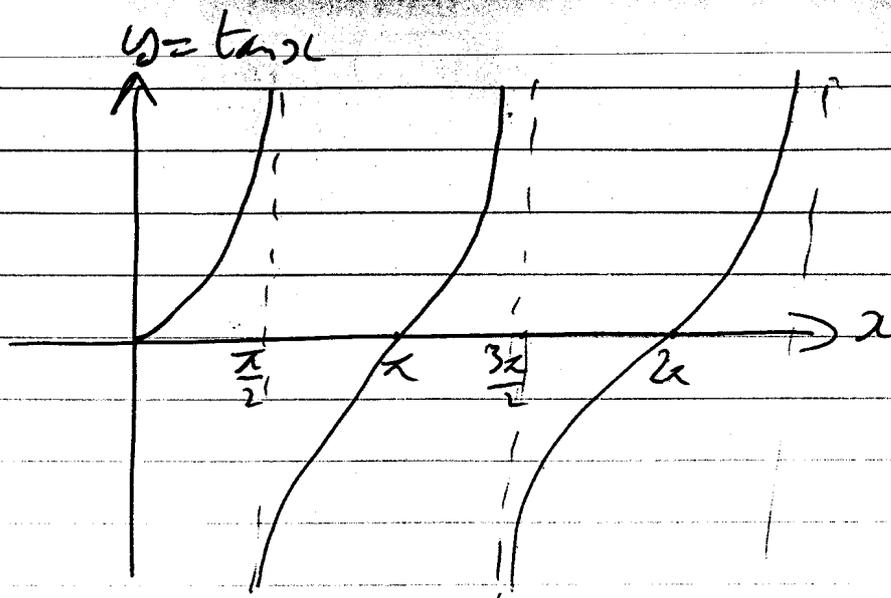
$$(ka)^2 + (qa)^2 = \frac{2ma^2 V_0}{\hbar^2}$$

We also have

$$k \cot ka = -q.$$

which we can rewrite as  $k \cot ka = -qa$

$$\text{Let} \quad y = qa \quad \alpha \quad x = ka$$



There are precisely 2 bound states if the circle corresponding to  $x^2 + y^2 = \frac{2ma^2V_0}{\hbar^2}$

crosses  $y = \pi \cot x$  only in the first two branches. If it only crosses the first branch there will only be one bound state

hence. The circle must cross the x-axis in the range,

$$\frac{3\pi}{2} < x < \frac{5\pi}{2}$$

$$\Rightarrow \frac{3\pi}{2} < ka < \frac{5\pi}{2}$$

~~3~~

But for  $y = 0$   $(ka)^2 = \frac{2ma^2V_0}{\hbar^2}$

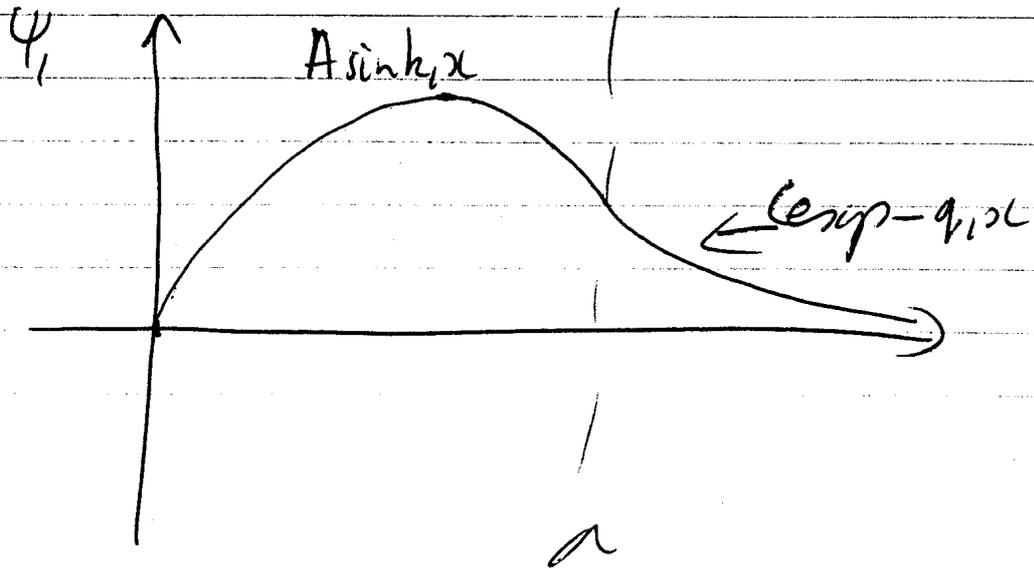
hence  $\frac{9\pi^2}{4} < \frac{2ma^2V_0}{\hbar^2} < \frac{25\pi^2}{4}$

$\Rightarrow$

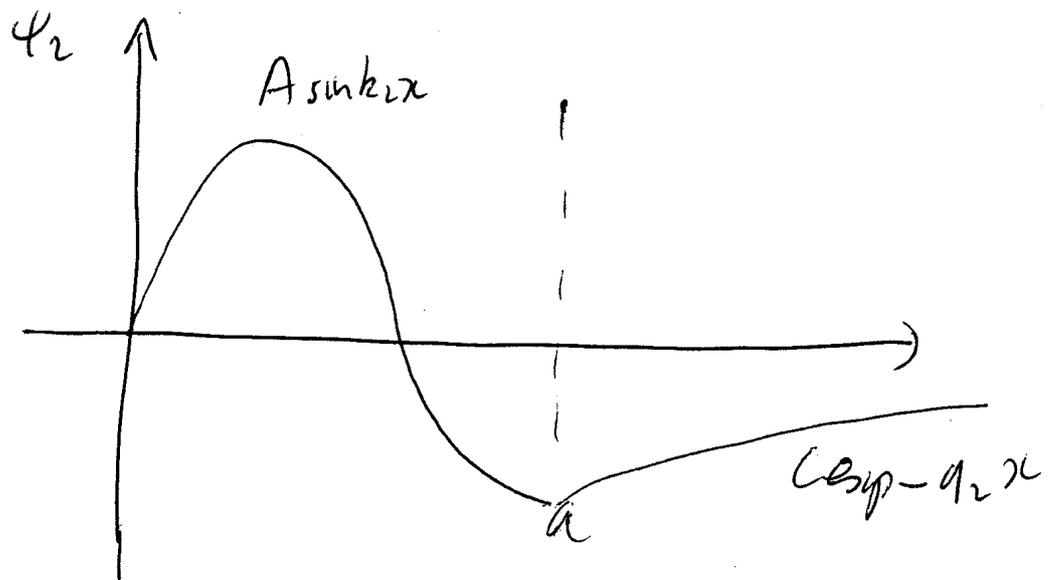
$$9 < \frac{8ma^2V_0}{\hbar^2 \pi^2} < 25$$



If  $\beta$  is close to but larger than 1  
 The wavefunction for the lowest state has the  
 form:-



The next state has the form.



Note  $q_2$  is much smaller than  $q_1$ , since  
 $k$  is close to  $\frac{3\pi}{2}$   $\alpha$   $(\frac{\hbar^2 k^2}{2m}) = \beta$