

2001

Paper 2

NATSCI: Part 1B, Part II (General)

Tues 29 May 2001

Advanced Physics (2)

B6

Electron wavefunction Ψ of a hydrogenic system

$$\Psi(r, \theta, \phi) = Y_{l,m}(\theta, \phi) R_{nl}(r)$$

↑
*N.B. misprint
in exam sheet

$Y_{l,m}(\theta, \phi)$: Spherical harmonics.

Give angular dependence of Ψ at fixed radius.

The set of functions labelled by l, m are common to all systems with spherically symmetric potentials.

l = quantum number of orbital angular momentum.

m = quantum number for component of orbital angular momentum along the quantization axis (z -axis usually).

$$\text{Total ang. mom (orbital)} = \sqrt{l(l+1)}$$

$$\text{component along } z\text{-axis} = \hbar m$$

l can take $0, 1, 2, 3, \dots$

m can be only $m = -l, -l+1, \dots, l$

$R_{nl}(r)$: Radial dependence of ψ .

n is the principal quantum number, which determines the electron energy.

$n' = n-1 \geq 0$ is the number of nodes of $R_{nl}(r)$.

$n = 1, 2, 3, \dots$ label atomic "shells" of states the electron can occupy.

Verify eigenfunction:

$$\psi = \frac{2}{\sqrt{4\pi}} \alpha^{3/2} e^{-\alpha r}, \quad \alpha \equiv \frac{Z}{a_0}.$$

Sch. eqⁿ:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(r) \right) \psi(r) = E \psi(r)$$

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \partial_r r^2 \partial_r \psi - \frac{Ze^2}{4\pi\epsilon_0 r} \psi = E \psi$$

$$\Rightarrow -\frac{\hbar^2}{2m} \left(\alpha^2 - \frac{2\alpha}{r} \right) - \frac{Ze^2}{4\pi\epsilon_0 r} = E, \quad \forall r$$

$$\Rightarrow E = -\frac{\hbar^2 \alpha^2}{2m} \quad \text{and} \quad \alpha = \frac{Ze^2 m}{4\pi\epsilon_0 \hbar^2}$$

$\therefore \Psi(r)$ is a solution of time-indep Sch. eq'n and the binding energy is

$$-E = \frac{\hbar^2 \alpha^2}{2m} = \frac{1}{2} m \left(\frac{2e^2}{4\pi\epsilon_0 \hbar} \right)^2$$

Helium atom ionized to He^+

Ionization energy = Binding energy of hydrogenic system with $Z=2$.

$$\begin{aligned} \therefore \text{I.E.} &= \frac{1}{2} m \left(\frac{2e^2}{4\pi\epsilon_0 \hbar} \right)^2 = 2m \left(\frac{e^2}{4\pi\epsilon_0 \hbar} \right)^2 \\ &= \frac{2e^3 m}{(4\pi\epsilon_0 \hbar)^2} \text{ eV} \\ &= 54.4 \text{ eV} \end{aligned}$$

Most prob. distance = r such that $p(r)$ is max,

where $p(r)$ is probability density per unit radius.

$$P(r)dr = |\Psi|^2 r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi r^2 |\Psi|^2 dr$$

$$\therefore \max P(r) \text{ is at } r = a_0/Z = a_0/2 = 0.026 \text{ nm}$$

(B10)

See answers to 2002 P2 Q6 for full proof.

$$\text{phase velocity } v_p = \frac{\omega}{k}$$

$$\text{if } v_p \propto \omega^n \text{ i.e. } v_p = \beta \omega^n$$

then the dispersion curve can be written as

$$\omega = \beta \omega^n k$$

$$\Rightarrow \omega^{1-n} = \beta k$$

$$\text{differentiating } (1-n) \omega^{-n} \frac{d\omega}{dk} = \beta$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{\beta \omega^n}{1-n}$$

$$\text{But } v_g = \frac{d\omega}{dk} \Rightarrow \boxed{v_g = \frac{v_p}{1-n}}$$

Velocity has dimensions

$$[v_p] = \cancel{M} L T^{-1}$$

So we expect a relation of the form :-

$$[v_p] = \beta [\text{density}]^x [g]^y [\omega]^z$$

$$L T^{-1} = \beta M^x L^{-3x} L^y T^{-2y} T^{-z}$$

$$\begin{aligned} \text{equate powers for } M &: x = 0 & T: -1 = -2y - z \\ L &: 1 = -3x + y \end{aligned}$$

$$so \quad x=0, \Rightarrow y=1$$

$$1 = 2x_1 + z$$

$$z = -1$$

$$so \quad v_p = \beta \frac{g}{\omega} \quad \text{and } \omega \text{ independent of } p$$

$$\begin{aligned} \text{Period} &= 8s \Rightarrow \omega = \frac{2\pi}{T} \\ &= \frac{2\pi}{8} \\ &= \frac{\pi}{4} \end{aligned}$$

$$\lambda = 100m \Rightarrow 2\pi/\lambda = \pi/50$$

The dispersion relation is :-

$$\omega = \frac{\beta g}{\omega} k$$

$$\Rightarrow k = \frac{\omega^2}{\beta g}$$

$$\Rightarrow \beta = \frac{\pi^2}{16} / 9.81 \times \frac{\pi}{50} = \frac{50}{16} \frac{\pi}{9.81}$$

$$= \cancel{4.77} \quad 1 \quad !!$$

$$ie = 1.0008$$

$$v = f \lambda = \frac{\lambda}{T} = \frac{100m}{8s} = 12.5 \text{ m/s}$$

So the dispersion relation is

$$k = \sqrt{\omega^2/g} \Rightarrow v_p \propto \frac{1}{\omega} \propto \omega^{-1}$$

So the group velocity

$$v_g = \frac{v_p}{1-n}$$

$$= \frac{v_p}{1-(-1)}$$

$$= \frac{v_p}{2}$$

So the group velocity = ~~Ans~~ 6.2 ms^{-1}

$$\text{Day 1 period} = 12 \text{ seconds} = \frac{2\pi}{12} = \pi/6$$

$$3 \text{ days later period} = 8 \text{ s} \Rightarrow \omega = \pi/4$$

$$\text{phase velocity} = v_p = \frac{g}{\omega}$$

$$\text{So velocity for } T=12 \text{ s } v_p = \frac{9.81 \times 6}{\pi} = 18.7 \text{ ms}^{-1}$$

$$\text{for } T=8 \text{ s } v_p = \frac{9.81 \times 4}{\pi} = 12.3 \text{ ms}^{-1}$$

The waves travel a distance such that as we are looking at a wave with a single frequency we use the phase velocity.

The storm occurs at time t_0 before the first waves arrive

so for $T = 12s$

$$s = v_{12} t_0$$

for $T = 8s$

$$s = v_8 (t_0 + 3600 \times 3 \times 24)$$

$$s = v_8 (t_0 + t_{3\text{days}})$$

hence $s = v_8 \left(\frac{s}{v_{12}} + t_{3\text{days}} \right)$

$$s = \frac{s v_8}{v_{12}} + v_8 t_{3\text{days}}$$

$$\left(1 - \frac{v_8}{v_{12}}\right) s = v_8 t_{3\text{days}}$$

$$s = v_8 t_{3\text{days}} / \left(1 - \frac{v_8}{v_{12}}\right)$$

$$= \frac{12.5 \times 259200}{1 - \frac{12.5}{18.7}}$$

$$= 4772 \text{ km.}$$

$$t_0 = s/v_{12} = 6 \text{ days earlier.}$$