

Single Subject Physics

2000

Paper 1

(Q) (B). For first part see earlier question.

For plasma $c^2 k^2 = \omega^2 - \omega_p^2$

where $\omega_p^2 = \frac{Ne^2}{me\epsilon_0}$

@. phase velocity $v_p = \frac{\omega}{k}$

hence $v_p^2 = \frac{\omega^2}{k^2} = c^2 + \left(\frac{\omega_p}{k}\right)^2$

$$\Rightarrow v_p = \sqrt{c^2 + \left(\frac{\omega_p}{k}\right)^2}$$

group velocity $= \frac{d\omega}{dk} = v_g$

so $v_g \Rightarrow c^2 k = 2\omega \frac{d\omega}{dk}$

$$\Rightarrow \frac{d\omega}{dk} = c^2 \frac{k}{\omega}$$

$$\Rightarrow v_g \omega_p = \frac{\omega}{k} \frac{d\omega}{dk} = c^2$$

(1)

The phase velocity v_p

$$v_p = \cancel{f \cancel{c} \cancel{\omega}} \frac{\omega}{k}$$

$$c^2 k^2 = \omega^2 - \omega_p^2$$

$$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\frac{c^2}{v_p^2} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$v_p = c \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)^{-1/2}$$

$$\text{So for small } \omega_p \quad v_p \approx c \left(1 - \left(-\frac{1}{2}\right) \left(\frac{\omega_p}{\omega}\right)^2\right)$$

$$= c \left(1 + \frac{\omega_p^2}{2\omega^2}\right)$$

$$N = 3 \times 10^8 \text{ m}^{-3} \quad \text{so}$$

$$\omega_p^2 = \frac{3 \times 10^8 \times (1.6 \times 10^{-19})^2}{9.2 \times 10^{-31} \times 8.8 \times 10^{-12}}$$

$$\Rightarrow \omega_p = 9771 \text{ s}^{-1}$$

we know that

$$v_g v_p = c^2$$

$$\& \quad v_p \approx c \left(1 + \frac{1}{2} \left(\frac{\omega_p}{\omega} \right)^2 \right)$$

$$\text{so} \quad v_g \approx c \left(1 + \frac{1}{2} \left(\frac{\omega_p}{\omega} \right)^2 \right)^{-1}$$

$$\approx c \left(1 - \frac{1}{2} \left(\frac{\omega_p}{\omega} \right)^2 \right)$$

For 400 MHz ~~$v_g = c$~~

$$v_g - c = -c \times 2.98 \times 10^{-10} = \delta v_{400}$$

$$\text{For 200 MHz } v_g - c = -c \times 1.19 \times 10^{-9} = \delta v_{200}$$

So the pulses at 200 MHz arrive later because the ~~waves travel~~ group velocity is less.

The distance to the pulsar is

$$s = v_g (t_0 + t_{\text{delay}})$$

to the time for the 600 MHz pulses
to reach Earth

t_{delay} is the 2 days needed for the 200 MHz
pulses to arrive.

So for 600 MHz

$$s = v_{g_{600}} t_0$$

for 200 MHz

$$s = v_{g_{200}} (t_0 + t_{\text{delay}})$$

Hence

$$s = \frac{v_{g_{200}} s}{v_{g_{600}}} + t_{\text{delay}} v_{g_{200}}$$

$$s \left(\frac{v_{g_{600}} - v_{g_{200}}}{v_{g_{600}}} \right) = t_{\text{delay}} v_{g_{200}}$$

$$s = \frac{t_{\text{delay}} v_{g_{200}} v_{g_{600}}}{v_{g_{600}} - v_{g_{200}}}$$

$$S = \frac{t_{\text{delay}} (c - \delta v_{400})(c - \delta v_{200})}{(c - \delta v_{400} - c + \delta v_{200})}$$

$$= t_{\text{delay}} \frac{(c^2 - c(\delta v_{400} + \delta v_{200}))}{(\delta v_{200} - \delta v_{400})} \text{ to first order.}$$

$$\delta v_{400} + \delta v_{200} = c \times 1.5 \times 10^{-9}$$

$$\delta v_{200} - \delta v_{400} = c \times 8.9 \times 10^{-10}$$

$$S = t_{\text{delay}} \frac{(c^2 - c \times 1.5 \times 10^{-9})}{c \times 8.9 \times 10^{-10}}$$

$$= t_{\text{delay}} \times \cancel{c} \times 1.1 \times 10^4 c$$

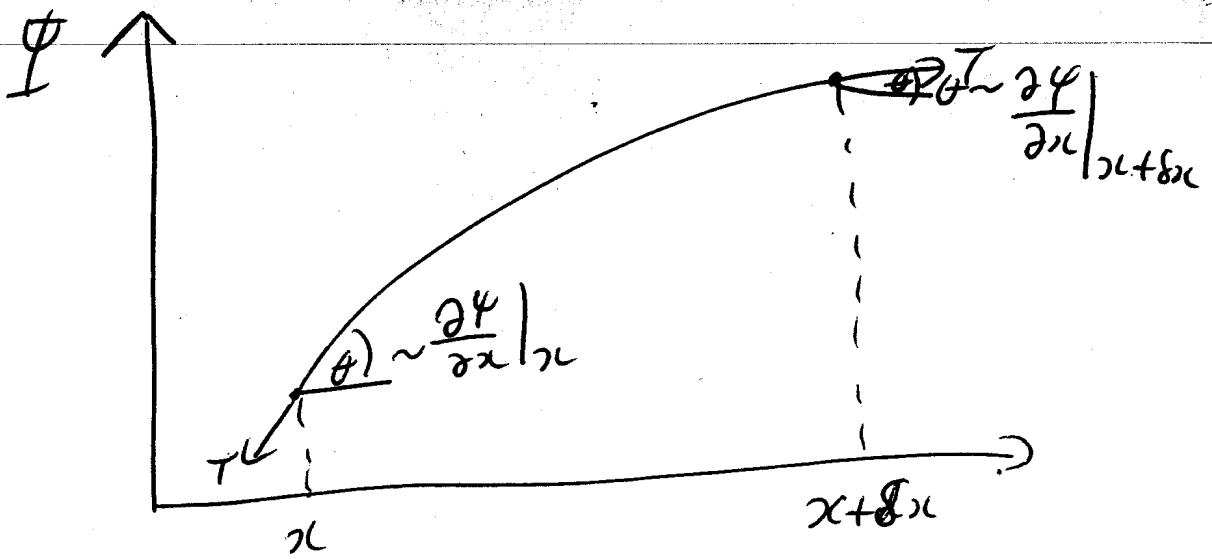
$$= \frac{2}{365.25} \times c \times 1.1 \times 10^9 \text{ light years}$$

$$= 6 \times 10^6 \text{ light years. ?}$$

Need to be careful with accuracy on these as differences from c are small.

4 seconds gives $1.7 \times 10^{18} \text{ m}$
 $\approx 1.7 \text{ light years}$

(B7.)



Consider a small element of the string of length δx & mass $\rho \delta x$ extending from x to $x + \delta x$

The restoring force at x is $T \sin \theta = T \frac{\partial \Phi}{\partial x}$

The restoring force at the other end is

$$-T \sin \theta = -T \frac{\partial \Phi}{\partial x} \Big|_{x+\delta x}$$

So the total force is

$$T \left(\frac{\partial \Phi}{\partial x} \Big|_x - \frac{\partial \Phi}{\partial x} \Big|_{x+\delta x} \right)$$

(restoring means force acting downwards, so a force acting up is negative)

$$= T \left(\frac{\partial \Phi}{\partial x} \Big|_x - \frac{1}{2} \left(\frac{\partial \Phi}{\partial x} \Big|_x + \frac{\partial^2 \Phi}{\partial x^2} \delta x \right) \right) = -T \frac{\partial^2 \Phi}{\partial x^2} \delta x$$

So the equation of motion is

$$-T \frac{\partial^2 \psi}{\partial x^2} \delta x = -\rho \delta x \frac{\partial^2 \psi}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -f \frac{\partial^2 \psi}{\partial t^2}$$

Which is the wave equation with

$$c = \sqrt{\frac{T}{\rho}}$$

Characteristic impedance defines the relationship between force and wave response it is given by:-

$$Z = \frac{\text{Driving force}}{\text{velocity response}}$$

For waves on a string the transverse driving force is

$$F = -T \frac{\partial \psi}{\partial x} \quad \text{transverse velocity} = \frac{\partial \psi}{\partial t}$$

$$\text{So } Z = - \frac{T \partial \psi / \partial x}{\partial \psi / \partial t}$$

Now for a wave $\psi = f(x - vt) = f(u)$

$$\frac{\partial \psi}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{df}{du}$$

And $\frac{\partial \psi}{\partial t} = \frac{ds}{du} \frac{\partial u}{\partial t} = v \frac{ds}{du}$

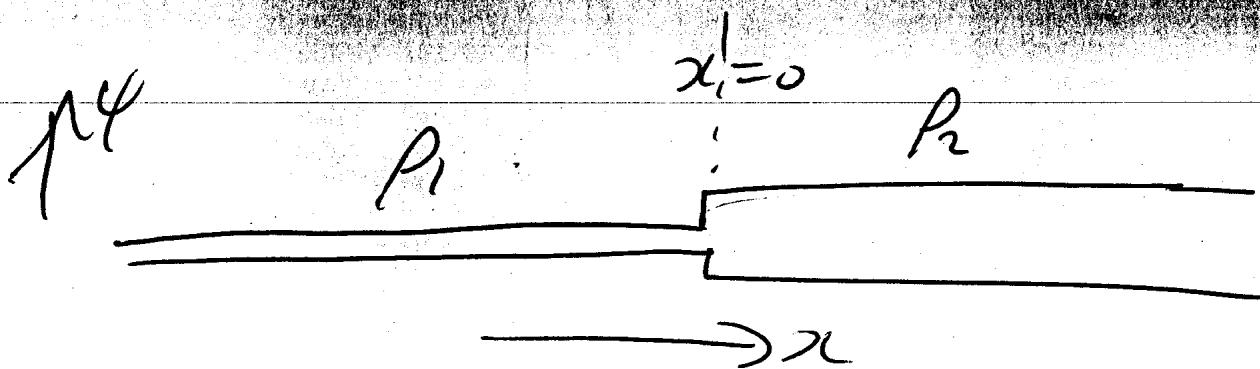
~~So~~ $\tau = T ds/du$

So $\frac{\partial \psi}{\partial x} = -v \frac{\partial \psi}{\partial t}$

$$\Rightarrow \tau = \frac{T}{v}$$

But $v = \sqrt{\frac{T}{\rho}}$

So $\tau = T / \sqrt{\frac{T}{\rho}}$
 $= \sqrt{\rho T}$



Boundary conditions for the wave :-

at $x = 0$

ψ cts

also force is continuous, so $T \frac{\partial \psi}{\partial x}$ is cts

$$\Rightarrow \frac{\partial \psi}{\partial x} \text{ cts.}$$

on LHS

$$\psi = A \exp i(\omega t - k_1 x) + R \exp i(\omega t + k_1 x)$$

$$\frac{d\psi}{dx} = -ik_1 (\exp i(\omega t - k_1 x) - R \exp i(\omega t + k_1 x))$$

on RHS

$$\psi = T \exp i(\omega t - k_2 x)$$

$$\frac{d\psi}{dx} = -ik_2 T \exp i(\omega t - k_2 x)$$

at $x=0$ applying boundary conditions

$$1+R = T$$

$$k_1(1-R) = k_2 T$$

So $\frac{k_1}{k_2}(1-R) = 1+R$

$\times \frac{k_1}{k_2} - 1 = \left(\frac{k_1}{k_2} + 1\right)R$

$$R = \frac{k_1/k_2 - 1}{k_1/k_2 + 1} = \frac{k_1 - k_2}{k_1 + k_2}$$

$\propto T = 1 + \frac{k_1/k_2 - 1}{k_1/k_2 + 1}$

$$= \frac{2k_1/k_2 + 1 - 1}{k_1/k_2 + 1}$$

$$= \frac{2k_1}{k_1 + k_2}$$

But $\omega = ck$ $\propto \omega = \text{constant}$

so $k = \frac{\omega}{c} = \omega \sqrt{\frac{F}{T}}$

So since T is constant.

$$R = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = \frac{\sqrt{\frac{\rho_1}{\rho_2}} - 1}{\sqrt{\frac{\rho_1}{\rho_2}} + 1}$$

$$T = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = \frac{2\sqrt{\rho_1/\rho_2}}{\sqrt{\rho_1/\rho_2} + 1}$$

Repeating the procedure.

$$t = \lambda/2$$



$$l = \frac{\lambda}{2} \Rightarrow k_l = \frac{2\pi}{\lambda} \frac{\lambda}{2} = \pi$$

In the middle region we have

$$\psi = A \exp(i(\omega t - k_2 x)) + B \exp(i(\omega t + k_2 x))$$

$$\frac{d\psi}{dx} = -ik_2 (A \exp(i(\omega t - k_2 x)) - B \exp(i(\omega t + k_2 x)))$$

In the third region

$$\psi = T \exp(i(lt - k_1 x))$$

$$\frac{d\psi}{dx} = -ik_1 T \exp(i(lt - k_1 x))$$

First region as before.

so at $x=0$ we have.

$$1 + R = A + B$$

$$k_1(1-R) = k_1(A - B)$$

at $x=l$ we have

$$A \exp(-ik_2 l) + B \exp(ik_2 l) = T \exp(-ik_1 l)$$

$$k_2(A \exp(-ik_1 l) - B \exp(ik_1 l)) = k_1 T \exp(-ik_1 l)$$

$$\text{now } \exp^{\pm ik_1 l} = \exp^{\pm ix} = -1$$

$$\text{so we have } -(A + B) = T \exp(-ik_1 l)$$

$$-k_2(A - B) = k_1 T \exp(-ik_1 l)$$

$$A+B = \frac{h_2}{k_1} (A-B)$$

But $I+R = A+B$

$$I-R = \frac{h_2}{k_1} (A-B)$$

so $2R = A+B - \frac{h_2}{k_1} (A-B)$

$$= 0.$$
