Four-vectors from the light clock

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Simple-minded physicist that I am, I can construct only the simplest of clocks: a light clock (see Figure 1).

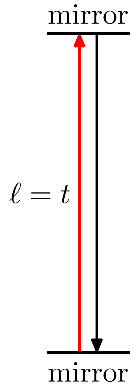

Figure 1. Light clock. This clock is a light beam bouncing vertically between two mirrors. The beam travels first (red line) from the bottom to the top mirror in time t, which is also the vertical separation ℓ in the convenient $c = 1$ units. The x separation of the mirrors is zero.

To an observer moving to the left at speed v, the clock looks like Figure 2.

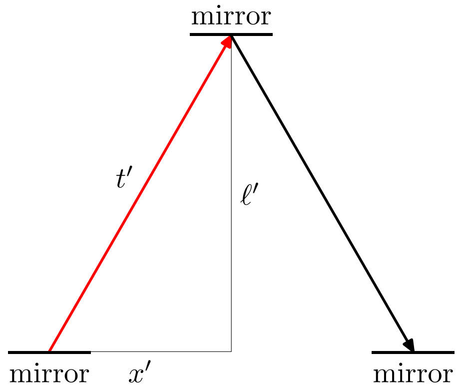

Figure 2. Moving light clock (primed frame). As judged by the moving observer, the beam travels from top to bottom (red line) in time t'. By the invariance of light speed, she sees the beam travel a distance t'. The distance x' is how far the top mirror travels in one bounce, again as seen by the moving observer. The moving observer judges the vertical separation to be ℓ'.

In the rest frame $x = 0$, and

$$t^2 - x^2 = \ell^2.$$

In the moving frame, Pythagoras says

$$t'^2 - x'^2 = \ell'^2.$$

Since lengths perpendicular to the direction of motion look the same in both frames (because of a sly symmetry argument that escapes me), the vertical lengths ℓ and ℓ' are equal. So the magic quantity

$$t^2 - x^2$$

is the same in both frames. It's invariant!