Spacetime diagrams

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Here are notes on spacetime diagrams and the magic of the interval. I wrote them by first making the diagrams, then creating captions as glue. I hope this method of writing a document works out. Comments welcome!

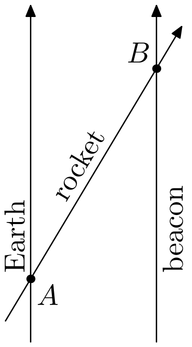

Figure 1. Worldlines in spacetime. In all spacetime diagrams, time goes up and space (the x-axis) goes right. The Earth sits in one spot, moving only through time: Its worldline is an upwards arrow. A beacon far from earth also sits in one spot, so its worldline is also an upwards arrow. A rocket passes earth (event A) heading for the beacon, which it eventually reaches (event B).

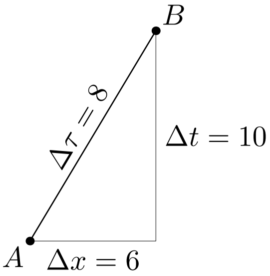

Figure 2. The interval. Any two events, here A and B as in the previous figure, have a space separation Δx and a time separation Δt. The combination

$$(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2$$

is magic. Even though different observers disagree about Δt and Δx – the barn paradox! – all observers agree on $\Delta\tau$, **the interval**. Let's say that the spaceship travels towards the beacon at $v = 3/5$ (always using the convenient $c = 1$ units), and the beacon is 6 light-years from earth (or, in $c = 1$ units, 6 years). Let's also choose a convenient unit of time, here the year. Then $\Delta t = 10$ and, to make the slope 5/3, $\Delta x = 10v = 6$, so

$$\Delta\tau = \sqrt{10^2 - 6^2} = 8.$$

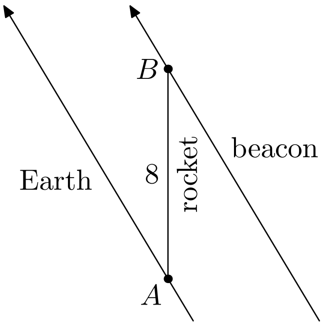

Figure 3. Interval as wristwatch time. The quantity $\Delta\tau$ is the same for all observers – so what? To see why we care, choose a particular reference frame, the rocket, and redraw the three worldlines. The rocket sees, for example, Earth moving backwards at speed v, hence the backwards sloping worldline for Earth. In this primed frame, the events A and B happen in the same place – in the rocket. So $\Delta x' = 0$ and $\Delta\tau$ (no prime needed because it's the same for everyone) is $\Delta t'$. Thus, $\Delta\tau$ is the time measured by an observer moving with the frame: the *wristwatch time* or the *proper time*. The rocket measures 8 years between A and B.

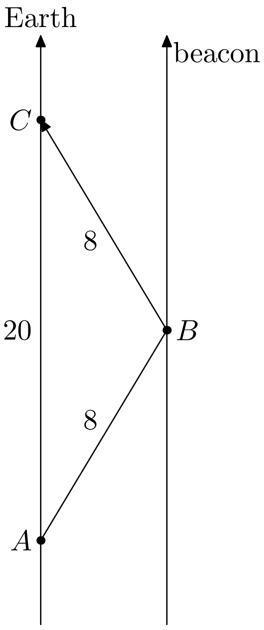

Figure 4. Twin paradox. Having reached the beacon (event B), the rocket turns around and heads towards Earth with speed $v = 3/5$. Its arrival home is event C. The wristwatch time going directly between A and C is 20 years. The wristwatch time going via B is 16 years. The straight paths has the longest wristwatch time! In regular, Euclidean geometry, the straight path is the shortest distance between two points. In spacetime (Lorentz geometry), the straight path is the longest distance between two events: all because of the minus sign in the definition of the magic interval.

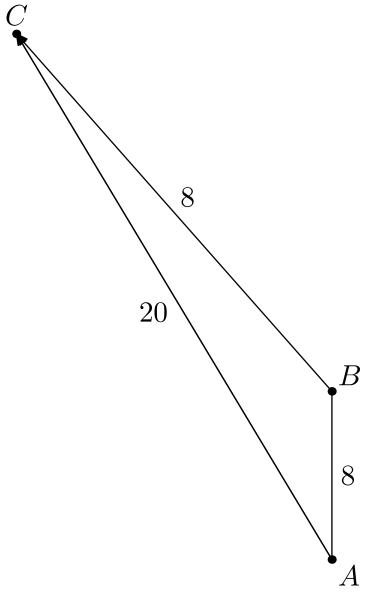

Figure 5. Twin paradox from outgoing frame. In the frame moving with the outgoing rocket, the intervals are as in the Earth frame: AB is 8 years, BC is 8 more years, and AC (directly) is 20 years. This diagram is Figure 3 but including the return journey (BC).

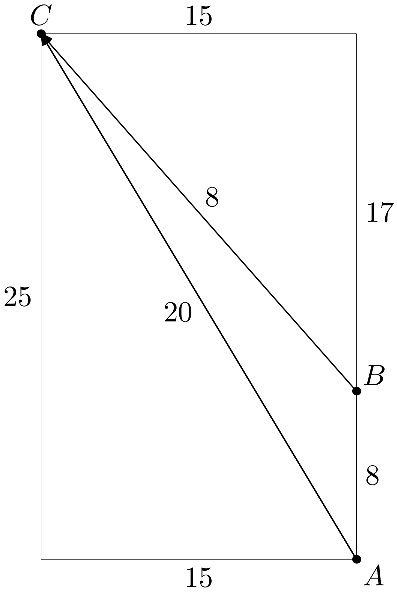

Figure 6. Calculating in the outgoing frame. The interval allows easy calculation in the outgoing frame. The interval from A to C is 20 years, and the worldline has slope $1/v = 5/3$. Solving for the $\Delta t'$ and $\Delta x'$ gives 25 and 15 years, respectively. Then the time difference (in this primed frame) between B and C must be $25 - 8 = 17$ years. So the returning rocket has speed $15/17$, as calculated in the outgoing frame. Is the following equality coincidence:

$$\frac{\frac{3}{5}+\frac{3}{5}}{1+\frac{3}{5}\frac{3}{5}} = \frac{15}{17}?$$

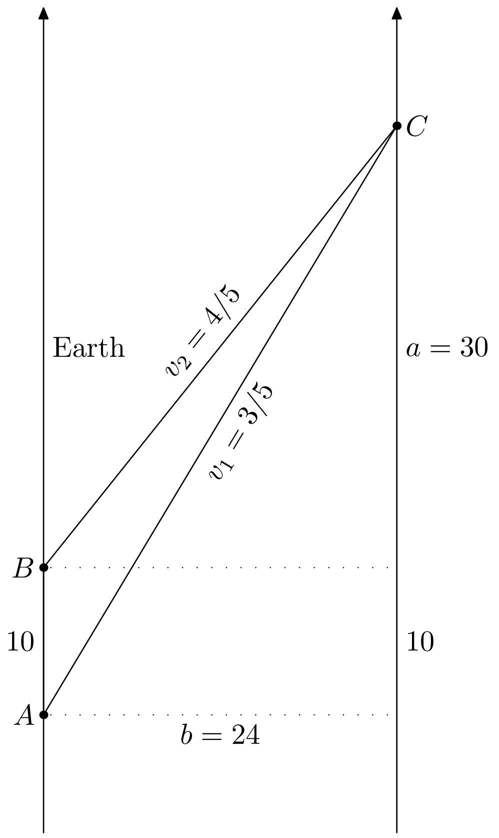

Figure 7. Twin paradox again! This diagram represents problem 21 from the Examples sheet. The first spaceship sets of (event A) at speed $v_1 = 3/5$. The second spaceship twiddles its thumbs for 10 years, then leaves (event B) at speed $v_2 = 4/5$. They meet at C. When and where is C? For the moment a and b are unknown. The triangle with the first rocket's worldline as hypotenuse gives one equation:

$$\frac{a+10}{b} = \frac{5}{3}.$$

The second rocket's worldline gives the second equation:

$$\frac{a}{b} = \frac{5}{4}.$$

The solution is $a = 30$ and $b = 24$. So the first hypotenuse has 'length' (wristwatch time)

$$\sqrt{40^2 - 24^2} = 8\sqrt{5^2 - 3^2} = 32$$

and the second hypotenuse has 'length' (wristwatch time)

$$\sqrt{30^2 - 24^2} = 6\sqrt{5^2 - 4^2} = 18.$$

The total wristwatch time for the second rocket, including the 10 years of twiddling its thumbs, is 28 years, which is 4 years shorter than that experienced by the first rocket. Once again, the straight path is longer!

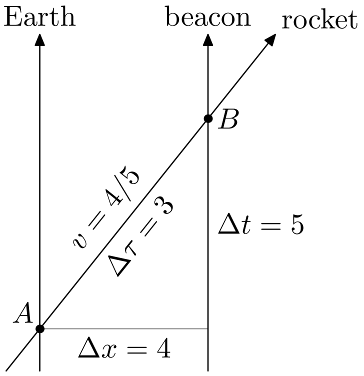

Figure 8. Problem 22 from the Examples sheet. A spaceship sets off from Earth (event A) at $v = 4/5$, at noon on both Earth and ship clocks. Using units of 10 minutes: 3 units later, by the ship's clock, it passes a beacon reading Earth time (event B). The slope of the worldline is 5/4 so $\Delta t/\Delta x = 5/4$; furthermore, $\Delta\tau = 3$ so $(\Delta t)^2 - (\Delta x)^2 = 9$. The solution is $\Delta t = 5$ and $\Delta x = 3$. So the beacon is 30 light-minutes (3 units) distant. When the ship passes it, its clock reads 12:50 (noon plus 5 units).

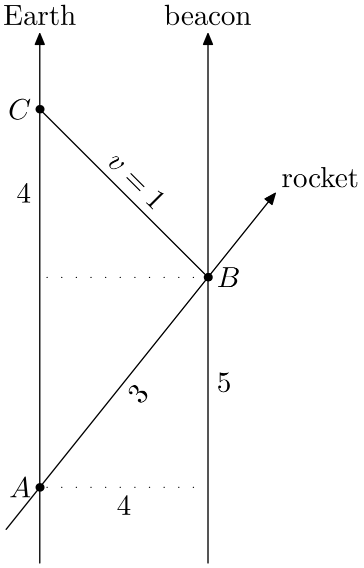

Figure 9. Ship signals Earth. The ship from problem 22 signals Earth when it passes the beacon and the signal arrives at Earth (event C). By Earth's clock, $5 + 4 = 9$ units pass: The time is 13:30.

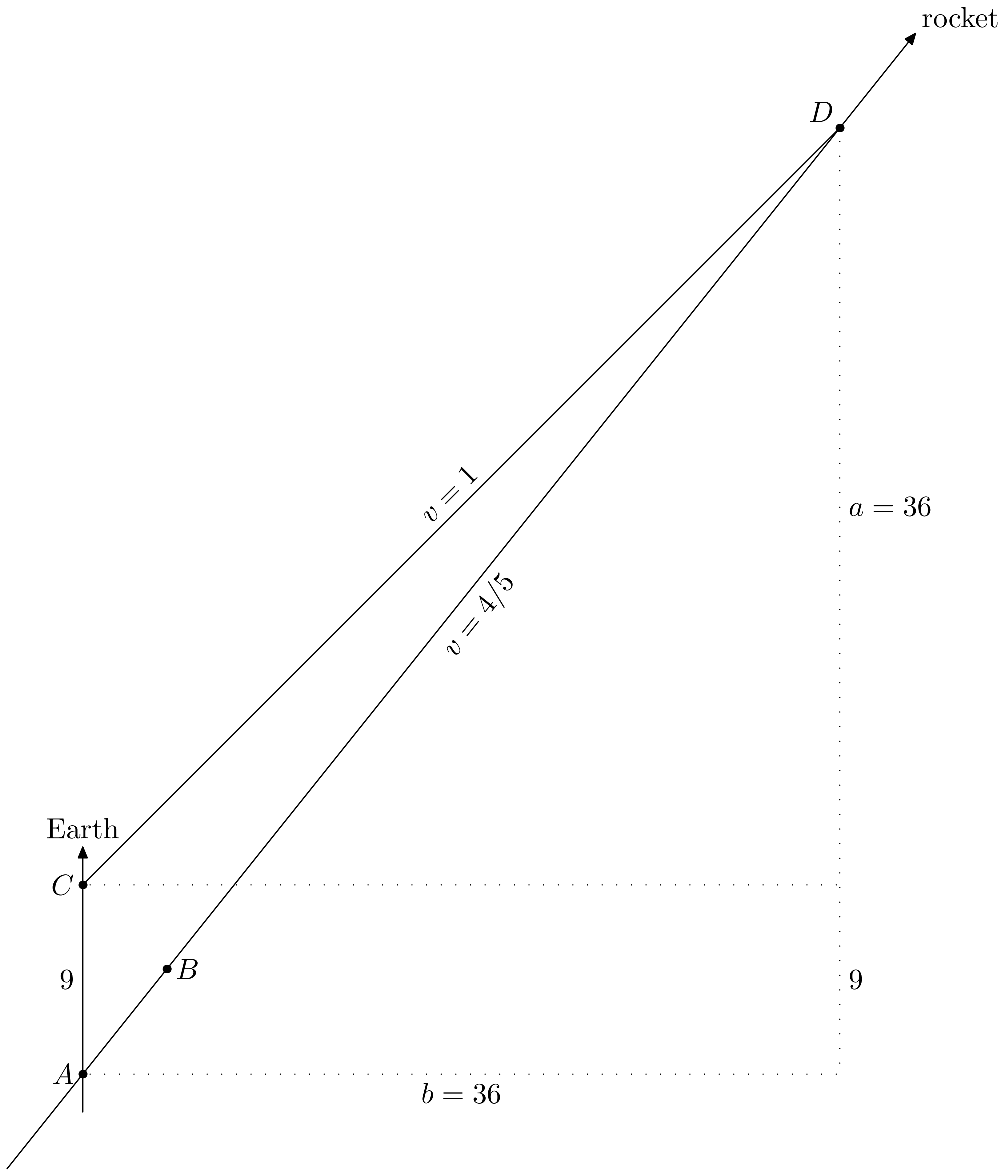

Figure 10. Earth signals back. Having received the signal from the ship, Earth signals back. That signal reaches the ship (event D). The paths AD and CD (light signal) give two equations, respectively:

$$\frac{a+9}{b} = \frac{5}{4}, \qquad \frac{a}{b} = 1.$$

Their solution is $a = b = 36$, which determine the wristwatch time (interval) between A and D:

$$\tau_{AB} = \sqrt{45^2 - 36^2} = 9\sqrt{5^2 - 4^2} = 27.$$

In normal units, 27 is 270 minutes, so the ship's wristwatch reads 16:30.

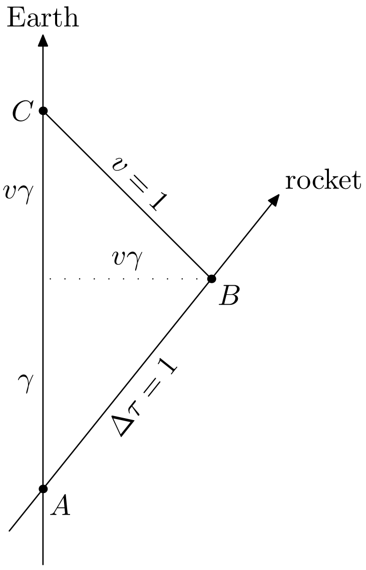

Figure 11. Doppler shift. A ship running from Earth sends back pulses once per unit time (choose your favorite unit). The first pulse is sent at B – after 1 unit of wristwatch time on the ship – and received on Earth at C. The Δx between A and B is $v\gamma$ and the Δt is γ (time dilation for free!). How do I know? Because those values satisfy two requirements: (1) the interval $\Delta\tau$ is 1, so $(\Delta t)^2 - (\Delta x)^2 = 1$, and (2) the speed is v, so $\Delta t/\Delta x = 1/v$. The solution is

$$\Delta t = \frac{1}{\sqrt{1-v^2}} \quad \text{and} \quad \Delta x = \frac{v}{\sqrt{1-v^2}}.$$

In relativity, the factor $1/\sqrt{1-v^2}$ shows up so often that you save lots of ink and thought by defining

$$\gamma \equiv \frac{1}{\sqrt{1-v^2}},$$

giving the expressions for Δt and Δx in the diagram.

The time between A and C, measured on Earth, is

$$\gamma(1+v) = \sqrt{\frac{(1+v)^2}{1-v^2}} = \sqrt{\frac{1+v}{1-v}}.$$

So the Earth observer sees pulses arrive at a slower rate: the Doppler shift, by the square-root factor above.

This diagram looks suspiciously like that of problem 22. There $v = 4/5$, so the Doppler factor is

$$\sqrt{\frac{1+\frac{4}{5}}{1-\frac{4}{5}}} = 3.$$

And look at that: waiting 3 units of wristwatch time on the ship (getting to B and sending a signal) meant that the signal was received at $3 \times 3 = 9$ units later on Earth. From the rocket's point of view, Earth runs away at $v = 4/5$, producing a Doppler factor of 3. A pulse leaves Earth after 9 units; with a Doppler factor of 3, the pulse reaches the ship after $9 \times 3 = 27$ units of ship time, as in Figure 10.