Problem set 6

Wednesday pairs please hand in to my pigeonhole by Wednesday 10am, Friday pairs by Thursday 1pm. Clearly explain your reasoning.

1 *Slipping and sliding*
You give a snooker ball (mass m and radius r) a horizontal impulse through its centre of mass and it starts to move with velocity v_0. Let μ be the coefficient of sliding friction.

a) At first, the ball skids; eventually, at some time t_0, it starts to roll. Why? On the same graph, sketch qualitatively the centre-of-mass velocity $v(t)$ and the scaled angular velocity $r\omega(t)$ (rather than ω, because ω and v do not have the same dimensions), label any interesting features, and explain your reasoning. Be sure to specify your sign convention for ω.

b) Qualitatively, how does t_0 depend on m, r, μ, v_0, and g? How should the mass distribution within the ball affect t_0? (For example, how should t_0 for a spherical-shell ball compare with t_0 for a solid-sphere ball?) Based on your qualitative reasoning, *guess* an expression for t_0. Make sure that your guess has dimensions of time!

c) Now analyse the motion quantitatively. Solve for $v(t)$ and $r\omega(t)$, and sketch them on the same graph. What is t_0? Compare with your guess in part b, and discuss any differences.

d) What is the final velocity of the ball, $v(t_0)$? What is the ball's kinetic energy? What fraction of its initial kinetic energy has it lost?

e) How much work is done by the force of sliding friction? Is your result consistent with the energy loss from part d?

f) Try it out: Strike a snooker ball as described, and collect whatever data you need to make a rough estimate of μ.

2 *Stirling's formula*
Stirling's formula says that, for large n,

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{2.1}$$

Here are two ways to derive a rough version of this formula.

a) The first version derives an expression for $\log n!$, which is also $\sum_{k=1}^{n} \log k$. Sketch a graph of $\log k$ and mark the area represented by the sum $\sum_{k=1}^{n} \log k$. As an approximation, replace the sum by an integral of $\log k$ and evaluate it to get an approximation to $\log n!$. Does the integral over- or underestimate the sum?

b) i) For the second method, begin with a useful trick: differentiating under the integral sign. You know that $\int_0^\infty e^{-t}\,dt = 1$ and, by changing variables, that

$$\int_0^\infty e^{-at}\,dt = \frac{1}{a}. \tag{2.2}$$

Now differentiate both sides of this expression n times with respect to a, and show that

$$\int_0^\infty t^n e^{-t}\,dt = n!. \tag{2.3}$$

ii) By approximating the integral (2.3), you can approximate $n!$. The integrand is also $e^{f(t)}$ where $f(t) = n \log t - t$. Sketch $f(t)$ as a function of t. Where is its maximum (call it t_0)? For large n, the exponential of $f(t)$ is even more sharply peaked than $f(t)$ itself; most of the contribution to the integral comes from around t_0. Therefore, $n! \sim e^{f(t_0)}$. What is the resulting approximation? How does it compare with Stirling's formula (2.1)?

iii) This last approximation, $n! \sim e^{f(t_0)}$, is dodgy: It neglects the width of the sharply peaked function $e^{f(t)}$. A more accurate approximation is:

$$n! \sim e^{f(t_0)} \times \text{width of peak}.$$

Why? Draw a picture to explain the argument. Estimate the width (there are many reasonable ways to make this estimate) and refine your estimate from ii. How does it compare with Stirling's formula? How could you improve the approximation yet further? If you feel adventurous, derive the $\sqrt{2\pi}$ factor.

3 *Random walks*

A confusing feature of a random walk is the presence of square roots: Why in a random walk does it take on the order of N^2 steps to move a distance N? Here is one way to understand this bizarre behaviour. Imagine a particle making a one-dimensional random walk: with equal probabilities, it moves one step either to the left or to the right. Let d_n be its position after n steps, with $d_0 = 0$. We shall study $\langle d_n^2 \rangle$, the expected value of d_n^2.

a) After 0 steps, the distribution of possible d_0 is simple: There is only one possibility, that the particle is at the origin. So $\langle d_0^2 \rangle = 0$. After 1 step, the particle is at either -1 or $+1$, with equal probabilities. So

$$\langle d_1^2 \rangle = \frac{1}{2} \left\{ (+1)^2 + (-1)^2 \right\} = 1.$$

Work out the probability distribution for the particle position after 2 steps, and from the distribution, work out $\langle d_2^2 \rangle$. Repeat for $\langle d_3^2 \rangle$ and $\langle d_4^2 \rangle$. Generalise the pattern: After N steps, what is the expected squared distance $\langle d_N^2 \rangle$? Harder: Prove your result.

b) What is $\sqrt{\langle d_N^2 \rangle}$? Therefore explain the behaviour mentioned in the introduction.

4 *Air molecules*

a) Estimate the mean free path, l, of air molecules at room temperature. This length is the step size in a random walk.

b) Roughly how fast does an air molecule move? Call the speed c.

c) What dimensions does a diffusion coefficient have? How can you combine c and l into a diffusion coefficient? Estimate the diffusion coefficient, D, for air molecules in air (this coefficient is called the *self-diffusion* coefficient of air). Estimate how long it would take an air molecule to diffuse across a room.

d) Fast pieces of fluid donate momentum to neighbouring slow pieces of fluid; so the fast pieces slow down, and the slow pieces speed up. The viscosity measures the ease with which the momentum diffuses. In air, momentum is diffused by particle motion directly: The particles carry their momentum with them, so viscosity arises from the same physics as does molecular diffusion. The viscosity of air should therefore be related to the diffusion coefficient D, which you estimated in part c. What are the dimensions of viscosity? How can you turn D into a viscosity? Therefore estimate the viscosity of air, and compare with reality. Why can't you use the same method to estimate the viscosity of water?