Lecture 1 (2004-01-16)

Estimate the temperature of the earth

Questions from the real world come messy. They don't come with labelled formulae saying 'Use me' or 'Apply me'. So the first step is to get yourself familiar with the physical system.

Why is the earth hot at all?

Why, for example, isn't the earth as cold as space, whatever that means? One meaning is the temperature of the microwave background radiation, which is 2.7 K. You suggested several reasons that the earth is much warmer:

- Radioactivity in the crust.
- Trapping of heat by the atmosphere.
- Sunlight!

The first one we'll talk about later in the course. The second, well, keep it in mind. The main effect is from sunlight. To see how it heats the earth, imagine a cold earth (0 K) and now the sun starts shining on it. The earth's surface will slowly warm up, and as it warms up it'll radiate energy. Eventually the radiated energy will match the received energy. This steady state sets the earth's surface temperature.

So how much does the earth radiate at a given temperature, T? First, what sort of quantity are we looking for? Is it an energy? Is it a force? No, it's a power: energy/time. The sun feeds energy in at a particular rate and the earth sends it to space at the same rate.

Radiated power versus temperature

How does the radiated energy/time (the power) depend on T? The honest derivation of the *blackbody-radiation* formula is a complicated affair and not possible within classical thermodynamics. You can read about it any textbook on sadistical mechanics. We use a simpler approach, one that works for finding many formulae: *dimensional analysis.* You can read a lot about dimensional analysis in my textbook at `<www.inference.phy.cam.ac.uk/sanjoy/>` but I just use a cheap version of it here. In Part II ('Order of magnitude physics') I'll tell you the whole story about dimensional analysis.

You suggested the quantities on which P depends:

Symbol	*What it is*	*Why include it*
T	Temperature	We want to find it
c	Speed of light	We're talking about radiation
k	Boltzmann's constant	Only way to convert temperature to energy
A	Area of surface	More surface means more radiation

The radiated power is proportional to the surface area, so a more useful quantity is P/A, which is the *flux* or F.

We stir these quantities to make a flux. So here are their dimensions, including the F, where θ means dimensions of temperature (most often given in units of Kelvin degrees):

Symbol	*Dimensions*
F	MT^{-3}
T	θ
c	LT^{-1}
k	$\mathrm{ML}^2\mathrm{T}^{-2}\theta^{-1}$

So k and T must appear together or nowhere, since they are the only two quantities with temperature units θ. And kT and F are the only two quantities with a mass; since each has power of mass:

$$F \propto kT.$$

The constant of proportionality has dimensions of T^{-1}, to convert an energy into a power, but the only quantity left is c, which has a time but also a length. So there's no way to make the dimensions match. Alas!

We must have forgotten a quantity. This law cannot be derived from classical thermodynamics. Light comes in photons, and photon energies depend on $\hbar$. Including this quantity, the list is (after combining k and T):

Symbol	*Dimensions*
F	MT^{-3}
c	LT^{-1}
kT	$\mathrm{ML}^2\mathrm{T}^{-2}$
$\hbar$	$\mathrm{ML}^2\mathrm{T}^{-1}$

A cheap trick is to rewrite the list in terms of E, the dimensions of energy (i.e. $\mathrm{ML}^2\mathrm{T}^{-2}$), and use c^2 instead of c:

Symbol	*Dimensions*
F	$\mathrm{EL}^{-2}\mathrm{T}^{-1}$
c^2	EM^{-1}
kT	E
$\hbar$	ET

By playing around you can find that

$$\frac{(kT)^4}{c^2\hbar^3}$$

has dimensions of flux, so

$$F \sim \frac{k^4}{c^2\hbar^3}\,T^4.$$

The missing dimensionless constant, which comes from integrating over phase space so throws in factors of 2π, turns out to be $\pi^2/60$, and

$$F = \frac{\pi^2}{60}\,\frac{k^4}{c^2\hbar^3}\,T^4.$$

The gaggle of constants

$$\frac{\pi^2}{60}\frac{k^4}{c^2\hbar^3}$$

is called the Stefan–Boltzmann constant, σ, and the Stefan–Boltzmann law is written

$$F = \sigma T^4,$$

where $\sigma \approx 6 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$.

Calculating the earth's surface temperature: First attempt

How much energy from the sun hits the earth's surface? That flux is $F_{\text{sun}} \sim 1.3\,\mathrm{kW\,m^{-2}}$, a quantity worth memorising. That's how much comes in. What comes in must go out, so $\sigma T^4 \sim F_{\text{sun}}$, or

$$T \sim \left(\frac{F_{\text{sun}}}{\sigma}\right)^{1/4} \sim \left(\frac{1.3 \times 10^3\,\mathrm{W\,m^{-2}}}{6 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}}\right)^{1/4}$$

Check the units: The right side will end up with just a temperature, so all is well. Then do the arithmetic:

$$T \sim (0.2 \times 10^{11})^{1/4} = \left(\frac{1}{50} \times 10^{12}\right)^{1/4}$$

The fourth-root of 50 is roughly $\sqrt{7}$, or 2.65, so the temperature is about $1000/2.65\,\mathrm{K}$ or 380 K. In familiar units that is about 100 °C. Hot, hot, hot! A quick note on how to evaluate 1000/2.65 without a calculator:

$$\frac{1000}{2.6} = \frac{1000}{2.5 + 5\%} \approx \frac{1000}{2.5} - 5\% = 400 - 5\% \approx 380.$$

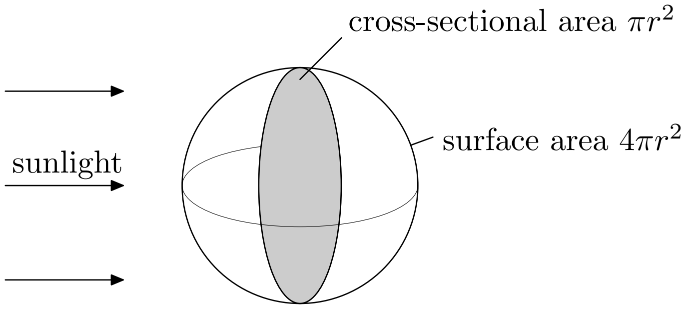

Figure 1. Flux hitting earth. The effective flux is one-fourth F_{sun} because only an area πr^2 intercepts the solar flux.

What went wrong so that the temperature is so high? The flux F_{sun} is for *perpendicular incidence*. But one-half of the earth is in darkness, and over the other half the angle of incidence varies; leaving aside the tilt of the earth's axis, only at the equator is the incidence perpendicular. The perpendicular area of the earth is πr^2, where r is the radius of the earth, and the power intercepted is spread over a surface area of $4\pi r^2$ (Figure 1). The effective flux is therefore $F_{\text{sun}}/4$. This factor is further reduced because some radiation bounces from or is absorbed in the atmosphere before reaching the ground. Roughly,

$$F_{\text{effective}} \sim F_{\text{sun}}/5.$$

So the surface temperature is reduced by a factor of $5^{1/4}$ – no need to recalculate T from scratch using σ because we instead can *scale* the previous result. A cheap trick for the fourth root of 5:

$$5 = \frac{80}{16} \approx \frac{81}{16} = \frac{3^4}{2^4}$$

so

$$5^{1/4} \approx \frac{3}{2}.$$

The new temperature estimate is then

$$\frac{380\,\text{K}}{3/2} \approx 255\,\text{K}.$$

In familiar units, it is $-18\,^\circ\text{C}$. Cold, cold, cold!

What went wrong? We neglected the trapping of heat by the atmosphere: the *greenhouse effect*. The ground absorbs sunlight and radiates it as infrared. The atmosphere contains CO_2 and water vapour, which are particular good at blocking infrared. So the infrared radiation has a hard time leaving. It's as if it goes slower than c. Of course it moves at the speed of light but the bouncing back and forth slows its effective speed. Thus the radiated power is less than σT^4, it turns out by a factor of 1.5, and the surface temperature goes up to compensate for this greenhouse reduction.

References and further reading

Blackbody radiation.
Adkins, pp. 94–97; Baierlein, pp. 123–125.

Stefan–Boltzmann law.
Adkins, pp. 97–99; Baierlein, pp. 123–125.

Dimensional analysis.
Textbook on *Order of magnitude physics* at `<www.inference.phy.cam.ac.uk/sanjoy/>`.