

Temperature in the atmosphere (from Sheet 3, question B1)

Why is an isothermal atmosphere unstable?

Let's say the temperature starts out constant. The pressure drops with height (the Boltzmann factor argument from IA physics), so as a parcel rises, it expands. As it expands, it cools and will be cooler than the surrounding air. When it and the surrounding air come to equilibrium, the temperature at that height will have dropped slightly. This process will continue until the surrounding temperature becomes the same as the air parcel's temperature, which is the temperature an adiabatic expansion would produce.

What is the temperature versus height?

There are two unknown functions: the pressure $p(z)$ and the temperature $T(z)$. One equation connecting them comes from the ideal gas law, which applies whether the atmosphere is adiabatic or isothermal:

$$p = nkT.$$

The other comes from the adiabatic law:

$$pV^\gamma = \text{const.}$$

Damn, now I have a third unknown, the volume. But I remove it by rewriting the ideal gas law as

$$pV = NkT,$$

where $N = nV$ is the number of molecules rather than their number density. Now I rewrite the adiabatic law to contain the product pV . The first step is to take the γ 'th root of both sides,

$$p^{1/\gamma}V = \text{const.},$$

and then extract a pV :

$$p^{-1+1/\gamma}pV = \text{const.}$$

The ideal gas law gives a replacement for pV :

$$p^{-1+1/\gamma}T = \text{const.},$$

where the constants N and k have been slurped into the right hand side. So

$$T \propto p^{1-1/\gamma}. \quad (1)$$

Two equations, the ideal gas law and the adiabatic law, have turned into one equation. Try to work out why that happened. No matter the reason, I'm still one equation short. Equation (1) determines the temperature given the pressure. The missing constant of proportionality is determined by the conditions, say, at sea level: $p = 1 \text{ atm}$ and $T \sim 18^\circ\text{C}$ (or whatever it is). But even with that constant, (1) is not the full story because I do not know how the pressure changes with height. The pressure, therefore, must be determined by another bit of physics.

New bit of physics

To find the new physics, return to a simpler case: the isothermal atmosphere from IA physics. There the pressure drops because less fluid lies above you as you go higher. In other words, it drops because of gravity, which I now include.

Imagine a thin slab of air at height z with area A and thickness Δz . The pressure at the bottom produces an upwards force $Ap(z)$. The pressure at the top produces a downwards force $Ap(z + \Delta z)$. The net effect is a downwards force $A\Delta p$, where

$$\Delta p = p(z + \Delta z) - p(z).$$

Here Δp will be negative, since pressure decreases with height, so the force is a negative downwards force and therefore a positive upwards force. Gravity provides a downwards force $mg = \rho g A \Delta z$. It balances the pressure resultant so $-A\Delta p = \rho g A \Delta z$. The area cancels, which is a useful check, and

$$\Delta p = -\rho g \Delta z.$$

Check the signs!

Damn, this equation introduces a new unknown, the density ρ . Fortunately the ideal-gas law connects density and pressure: $p = nkT$ so $mp = mnkT$, where m is the mass of one ‘molecule’ of air. Since $\rho = mn$:

$$\rho = \frac{p}{kT}.$$

The gravity equation then becomes

$$\Delta p = -\frac{mg}{kT} p \Delta z.$$

This equation looks cleaner as a fractional change:

$$\frac{\Delta p}{p} = -\frac{mg}{kT} \Delta z. \tag{2}$$

Combining the equations

Equation (1) will provide another expression for $\Delta p/p$. If $f = x^n$, then $\Delta f/f = n(\Delta x/x)$. So

$$\frac{\Delta T}{T} = \left(1 - \frac{1}{\gamma}\right) \frac{\Delta p}{p}.$$

To get shorter equations, I define

$$\beta \equiv 1 - \frac{1}{\gamma},$$

so

$$\frac{\Delta T}{T} = \beta \frac{\Delta p}{p}.$$

I chose $\beta = 1 - 1/\gamma$ rather than its negative because that choice makes β positive. Then

$$\frac{\Delta p}{p} = \frac{1}{\beta} \frac{\Delta T}{T}.$$

Put this equation into the gravity result (2):

$$\frac{\Delta T}{T} = -\beta \frac{mg}{kT} \Delta z.$$

Bonus: The temperature cancels, so

$$\frac{\Delta T}{\Delta z} = -\beta \frac{mg}{k}.$$

The right side is only constants, so the temperature drops linearly with height. A simple relation!

Putting in numbers

To put in numbers, first multiply by $\Delta z/T_0$ where T_0 is the sea-level temperature:

$$\frac{\Delta T}{T_0} = -\beta \frac{mg}{kT_0} \Delta z.$$

As long as the dimensions are correct (check!), both sides are dimensionless. The left side is fine. For the right side to be dimensionless, mg/kT_0 must be an inverse height (since β is already dimensionless). Thus kT_0/mg is a height; call it H_0 :

$$H_0 \equiv \frac{kT_0}{mg}.$$

Then

$$\frac{\Delta T}{T_0} = -\beta \frac{\Delta z}{H_0}. \quad (3)$$

To evaluate H_0 , multiply by N_A/N_A :

$$H_0 = \frac{N_A k T_0}{N_A m g} = \frac{R T_0}{M g},$$

where M is the molar mass of air, so

$$H_0 \sim \frac{8 \text{ J mol}^{-1} \text{ K}^{-1} \times 300 \text{ K}}{0.03 \text{ kg mol}^{-1} \times 10 \text{ m s}^{-2}} \sim 8 \text{ km}.$$

For dry air, $\gamma = 1.4$ so $\beta \approx 0.3$. Put it all into (3):

$$\frac{\Delta T}{T_0} \sim -0.3 \times \frac{\Delta z}{8 \text{ km}}.$$

For a 1 km mountain (e.g. Snowdon) that gives $\Delta T \sim -10^\circ\text{C}$. Which is quite reasonable. A check on the Internet showed a graph of Snowdon's year-round temperature and it values from 4 to 14 $^\circ\text{C}$, about 10 $^\circ\text{C}$ colder than sea level.