Here are some solutions to the revision problems. Also study the example sheets and solutions, as well as the examples done in lectures.

1 *Sound waves*

Sound waves have compressed regions alternating with rareified regions. If the compression is fast, so heat has no time to diffuse from the compressed to the rareified region, then the sound waves are adiabatic. If they are slow, so heat has time to diffuse, then they are isothermal. To decide, compare two time scales: the time *available* for heat to diffuse, and the time *required* for heat to diffuse.

Imagine a sound wave with frequency f. The time available for heat to diffuse is, except for a dimensionless constant, $1/f$: In one cycle, the rareified regions have become the compressed regions and back again, so to make the wave isothermal the heat had better arrive faster than that.

How long does heat require to diffuse? The sound wave has wavelength $\lambda = c/f$, where c is the speed of sound. Compressed areas are separated from rareified areas by a substantial fraction of λ. So, except for dimensionless constants, heat has to diffuse a distance λ, which takes time

$$\tau_{\text{diffuse}} \sim \frac{\lambda^2}{\kappa} = \frac{c^2}{f^2\kappa},$$

where κ is the thermal diffusivity. The time available is $\tau_{\text{avail}} \sim 1/f$, so the time ratio is:

$$R \equiv \frac{\tau_{\text{avail}}}{\tau_{\text{diffuse}}} \sim \frac{f\kappa}{c^2}.$$

If $R \gg 1$ the waves are isothermal; if $R \ll 1$ the waves are adiabatic. Having f in the numerator make sense. True, higher frequency waves provide less time for heat to diffuse (the heat has to diffuse before the wave moves on), but the distance is also smaller; diffusion times are quadratic in distance, so the benefit of the smaller distance more than compensates for the shorter time available. Hence high-frequency waves tends towards being isothermal.

Now I'll evaluate the critical frequency where the crossover to adiabatic happens, I can simplify the ratio because $\kappa \sim c\ell$, where ℓ is the mean free path of the air molecules (Lecture 4). So $R \sim f\ell/c$. The ratio ℓ/c is the the average time between collisions for an air molecule. So c/ℓ is the collision frequency for an air molecule and $R \sim f/f_{\text{collision}}$.

For air, $\ell \sim 10^{-7}$ m and $c \sim 300\,\text{m}\,\text{s}^{-1}$, so

$$f_{\text{collision}} \sim \frac{300\,\text{m}\,\text{s}^{-1}}{10^{-7}\,\text{m}} \sim 3 \times 10^9\,\text{Hz}.$$

A typical sound wave (middle C) has $f \sim 300$ Hz, so $R \sim 10^{-7}$. Ordinary sound waves are *very* adiabatic! For sound waves to be isothermal, f would have to be roughly the collision frequency, whereupon how could the sound travel (if the wave has to oscillate faster than than air molecules can bounce)?

Historical note: The first calculation of the speed of sound was due to Newton, who gave $c = \sqrt{p/\rho}$, where p is atmospheric pressure. This value, which assumes the compressions are isothermal, is low by 10 or 15 per cent, which puzzled Newton. Because sound waves are adiabatic, the compressed areas get hotter than if the waves were isothermal. Higher temperature means a higher pressure, so adiabatic compressions fight a stiffer spring than isothermal ones would. The result is that $c = \sqrt{\gamma p/\rho}$, where $\gamma = c_\text{p}/c_\text{v}$. For dry air, $\gamma = 1.4$, which would increase the speed by 20 per cent; wet air increases it by less.

2 *Breath*

Body-temperature air from your lungs can hold a lot of water vapor. As vapor, this water is invisible (it looks just like air: clear). When you exhale the air into the freezing winter air around you, the exhaled air cools. As it cools, it can no longer hold all that water vapor, which condenses into the tiny water droplets (fog) that you see.

3 *Freezing locks*

The expansion is quick (adiabatic) so $pV^\gamma = \text{const}$. Combining it with $pV = NkT$ gives

$$T \propto p^{1-1/\gamma}.$$

Here's how. Take the γ'th root of both sides of the adiabatic law:

$$p^{1/\gamma}V = \text{const},$$

and then extract a pV:

$$p^{-1+1/\gamma}pV = \text{const}.$$

The ideal gas law gives a replacement for pV:

$$p^{-1+1/\gamma}T = \text{const},$$

where the constants N and k have been slurped into the right hand side. So

$$T \propto p^{1-1/\gamma}. \tag{1}$$

Now I have to estimate γ for carbon dioxide. It is triatomic so it has 9 ($3N$) modes. It has the usual 3 translational modes. It is linear so it has 2 rotational modes, leaving 4 more. They include 2 bond-stretching modes and 2 bond-bending modes. The bond-stretching modes are probably frozen out at ordinary temperatures because bonds are very stiff against stretching. The bond-bending modes might be partially excited at ordinary temperatures; let's say that each is one-half excited or that one is fully excited. So c_v gets $(5/2)k$ from the translations and rotations and k from the bond-bending mode (why not just $k/2$?), making $\gamma \sim 9/7 \approx 1.3$, and $1 - 1/\gamma \approx 0.22$.

The pressure drops from 5 atm to 1 atm, so by a factor of 5. Therefore the temperature drops by a factor of $5^{0.22} \sim 1.4$ to $(300/1.4)\,\text{K} = 215\,\text{K}$ or $-60\,°\text{C}$. Which stress fractures the metal cycle lock, the better to hammer and shatter it.

4 *Turkey*

Cooking requires denaturing proteins. Usually we mean denaturing by heating. See Lecture 7 for more details about cooking and how heat denatures proteins.

In cooking a turkey, heat has to diffuse from outside in. How long that takes depends on the thickness. Doubling the mass but preserving the geometry means multiplying all lengths by $2^{1/3}$ or roughly $5/4$. Diffusion time is proportional to distance squared, so the diffusion (and hence cooking) time goes up by a factor of $25/16 \approx 1.55$. The small turkey cooked in 3 hours so this one takes roughly 4.6 hours.

5 *Tungsten*

For molecules to evaporate, they need to gain an energy l_{vap}, where l_{vap} is the heat of vaporization for one molecule. The Boltzmann factor then says that the vapor pressure is

$$p \propto e^{-l_{\text{vap}}/kT},$$

or

$$p \propto e^{-L_{\text{vap}}/RT},$$

where L_{vap} is the molar heat of vaporization. The problem gives p at 2 temperatures, so to get rid of the proportionality constant divide the pressures, then take the log:

$$\ln \frac{p_1}{p_2} = \frac{L_{\text{vap}}}{RT_2} - \frac{L_{\text{vap}}}{RT_1}.$$

So

$$L_{\text{vap}} = \frac{T_1 T_2}{T_1 - T_2} R \ln \frac{p_1}{p_2}.$$

For the given data (don't forget to conver to Kelvin):

$$\frac{T_1 T_2}{T_1 - T_2} = \frac{9818394\,\text{K}^2}{655\,\text{K}} = 1.5 \times 10^4\,\text{K},$$

so

$$L_{\text{vap}} \sim 1.5 \times 10^4\,\text{K} \times 8\,\text{J}\,\text{mol}^{-1}\,\text{K}^{-1} \times 6.9 \sim 0.8\,\text{MJ}\,\text{mol}^{-1}.$$

6 *Pluto*

Blackbody power flux is $F \propto T^4$. If Pluto is n times farther away from the sun than the earth is, then the arriving solar flux is smaller by a factor of n^2 compared to that at the earth's orbit. The blackbody temperature is therefore smaller by a factor of $n^{1/2}$. For Pluto $n \approx 40$ (you can look this one up or use Bode's law) so

$$T_{\text{Pluto}} \sim \frac{T_{\text{earth}}}{6} \sim \frac{288\,\text{K}}{6} \sim 50\,\text{K}.$$

The tables give 48 K. For more on blackbody temperatures see Lectures 1 and 2.

7 *Iron*

Each iron atom has three kinetic modes. Each atom also lives in a three-dimensional harmonic-oscillator potential due to the surrounding atoms, which provides three more modes. So by equipartition each atom contributes $3kT$ to the internal energy and $3k$ to the specific heat. The molar specific heat is $3R \sim 25\,\text{J}\,\text{mol}^{-1}\,\text{K}^{-1}$ (the law of Dulong and Petit). One mole of iron has mass 56 g, so the specific heat, given per mass, is $0.5\,\text{kJ}\,\text{kg}^{-1}$.

8 *Gases*

Helium is monoatomic, so it has three translational modes. By equipartition, the internal energy is $(3/2)RT$ per mole and $c_\mathrm{v} = (3/2)R$ per mole. For an ideal gas, $c_\mathrm{p} = c_\mathrm{v} + R$ so $c_\mathrm{p} = (5/2)R$. [You can do it per atom by changing R to k.]

Nitrogen is diatomic so it has three translational modes and two rotational modes. A diatomic molecule has six modes (from rearranging the six translational modes of two independent atoms), so nitrogen must have one vibrational mode. The bond is very stiff (it's a triple bond) so the vibration is frozen out at room temperature. Thus $c_\mathrm{v} = (5/2)R$ and $c_\mathrm{p} = (7/2)R$.

9 *Thermal conductivity*

See Lecture 5 for why $K = \rho c_\mathrm{p}\kappa$, and see Lectures 4 and 5 for the estimate of K for air.

10 *Conduction*

See Lecture 4 for a discussion of

$$F = K\frac{\Delta T}{\Delta x}.$$

11 *Lakes*

Let's say the air is at −25 °C and the 'warm' water underneath the ice is at 0 °C. Those temperature are the endpoint temperatures. But how does the temperature vary in between? Heat does not build up in the ice: The heat that enters at the bottom, from the water, flows through the ice and leaves at the top, into the air. So the flux must be constant through the ice. Remember the conduction equation: $F = K\Delta T/\Delta x$. So flux is proportional to temperature gradient. With the flux constant, the temperature gradient is also constant: The temperature profile is a straight line. As the ice thickens, the slope of the line changes, but it remains straight. Figure 1 has a sketch.

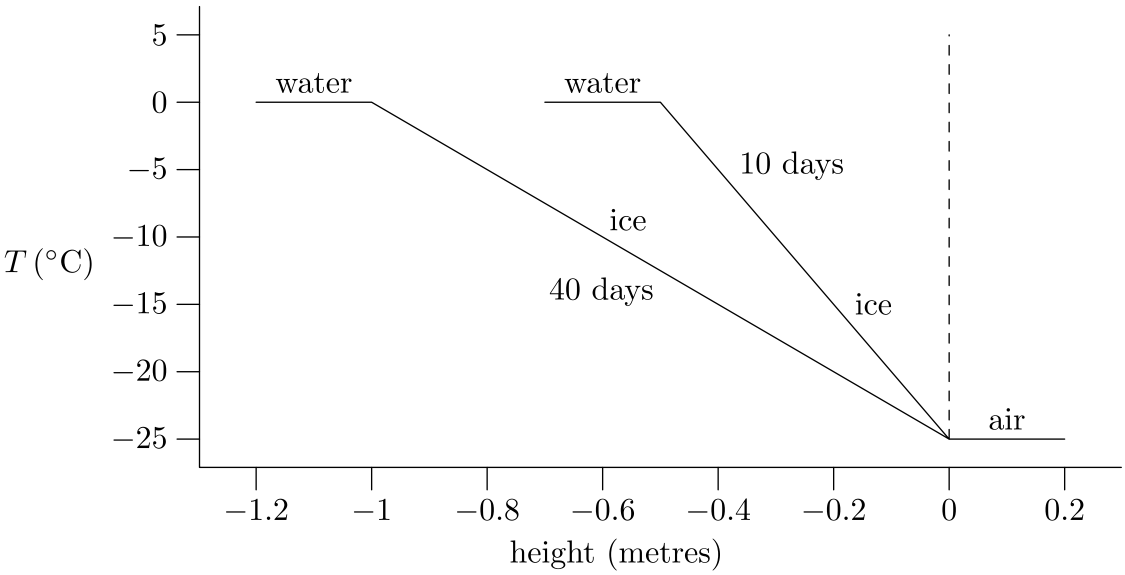

Figure 1. Temperature profiles. After 10 days the ice is roughly 0.5 m deep. After 40 days, the ice thickness has doubled, as you showed in the original solution (thickness grows as $\sqrt{t}$). So it is now 1 m deep. The temperature profile is still straight.

12 *Vaporisation*

See sheet 1, question 4 for several methods to estimate the heat of vaporisation.

13 *Sun's surface*

See Lecture 2 for several ways to work out the temperature.

14 *Size of molecules*

The mean free path ℓ is given by $n\sigma\ell \sim 1$, where n is number density and σ is the cross-section. The number density follows from the molar volume of 22 litres (at standard conditions):

$$n \sim \frac{6 \times 10^{23}}{2.2 \times 10^{-2}\,\mathrm{m}^3} \sim 3 \times 10^{25}\,\mathrm{m}^{-3}.$$

The cross-section is $\sigma = \pi d^2$, where d is the molecular diameter (why is it πd^2 rather than πr^2?). So,

$$d^2 \sim \frac{1}{\pi n \ell} \sim \frac{1}{3 \times 3 \times 10^{25}\,\mathrm{m}^{-3} \times 10^{-7}\,\mathrm{m}} \sim 10^{-19}\,\mathrm{m}^2.$$

So $d \sim 0.3\,\mathrm{nm}$ or $3.5\,\text{Å}$.

See sheet 2 (question A1) for the reverse problem: to find the mean free path of air molecules.

15 *Rain shadows*

I like Snowdon and it has a convenient height, 1 km. As an air parcels travels up Snowdon, the surrounding pressure drops and the parcel expands adiabatically, cooling in the process. If the atmosphere has reached convective equilibrium (become an adiabatic atmosphere), then the ambient temperature at the top of Snowdon will be the same as the cooler air-parcel temperature. Cooler air holds less water vapor; the excess vapor condenses and falls out as rain.

From Sheet 3, question B1 or from the similar derivation on the revision webpage, I find that temperature drops linearly with height and that it has dropped roughly 10 °C by the top of Snowdon. By what factor does the vapor pressure of water change when the temperature drops by 10 °C? The vapor pressure is given by

$$p \propto e^{-L_{\mathrm{vap}}/RT},$$

where L_{vap} is the molar heat of vaporization. For water, $L_{\mathrm{vap}} \sim 2\,\mathrm{MJ\,kg}^{-1}$; since 1 mol is 18 g, then $L_{\mathrm{vap}} \sim 40\,\mathrm{kJ\,mol}^{-1}$. For $T \sim 300\,\mathrm{K}$ the exponent is

$$-\frac{L_{\mathrm{vap}}}{RT} \approx -17.$$

Lowering T by 10 °C is a 3% change, so the exponent changes by 3% as well, or 0.5. Thus the vapor pressure drops by a factor of $e^{0.5} \sim 1.65$. If there was 1 unit of water vapor at sea level, there's $1/1.65 \sim 0.6$ left: so 40% of the water got dumped on the way up.

16 *Fogged windows*

Air inside the house is warm and can hold lots of water vapor. On a cold day, air near the window is colder; so when the room air goes near the window and cools, vapor condenses and deposits on the window. The outside air is already cold and therefore dry, so it does not deposit water on the outside of the window.

17 *Blanket*

Without the blanket, air easily flows past you. A parcel of air parks next to you for a bit, and you heat it up. Then it moves on – to be replaced by a new, cold parcel of air – and takes your heat with it. A thin layer of air remains next to you as your only protection. Why does a thin layer remain? Think of the dust layer on a fan blade: Even though air rushes past, a thin layer of dust sits undisturbed. Across this thin layer, heat is conducted. The flux is inversely proportional to distance, so you lose a lot of heat. To keep warm, you want a thicker layer of trapped air, which is what the blanket provides: It traps air in the spaces between the fibers.

Heat flux is $F = K\Delta T/\Delta x$, where K is the thermal conductivity of air, Δx is the thickness of the conduction layer (here the blanket or other barrier), and ΔT is the temperature difference across the layer. Imagine being in a cold room at night, say $0\,°\mathrm{C}$. Then with $K \sim 0.02\,\mathrm{W\,m^{-1}\,K^{-1}}$ and under a $2\,\mathrm{cm}$ blanket:

$$F = K\frac{\Delta T}{\Delta x} \sim 0.02\,\mathrm{W\,m^{-1}\,K^{-1}} \times \frac{30\,\mathrm{K}}{2\times 10^{-2}\,\mathrm{m}} \sim 30\,\mathrm{W\,m^{-2}}.$$

A typical person may have area $A \sim 2\,\mathrm{m} \times 1\,\mathrm{m}$ (front and back total width of $1\,\mathrm{m}$), so the power loss is $P = FA \sim 60\,\mathrm{W\,m^{-2}}$. You generate $100\,\mathrm{W}$ of heat:

$$P_{\text{metabolism}} \sim \frac{\text{daily food intake}}{1\text{ day}} \sim \frac{10\text{ jam donuts}}{10^5\,\mathrm{s}} \sim 10^2\,\mathrm{W},$$

since 1 jam donut is $1\,\mathrm{MJ}$. So a heat loss of $60\,\mathrm{W}$ is no problem.

However, in a thin T-shirt, say $2\,\mathrm{mm}$ thick, the heat loss goes up by a factor of 10 to $600\,\mathrm{W}$. Your body does not put out that much heat, so eventually your skin temperature drops (which reduces ΔT and therefore F) and then your core temperature drops. If it drops enough your enzymes fail and you die, so your body gets worried by drops in your core temperature; you therefore start to shiver as a way of generating more heat.

18 *Random walks*

For independent random variables, *their variances add.* Each step of the walk is random, so it adds zero to the expected location. But the variance of the location grows linearly with time because variances add. So the expected distance from the origin, which is the square root of the variance, grows as $\sqrt{t}$. See Lecture 4 for a more intuitive, pictorial explanation (basically, it explains why variances add for independent random variables).

19 *Gravity*

Gravity slingshots are based on conservation of energy and momentum – the same principles behind collisions of gas molecules with a container wall.

The solar system has many infinitely massive, moving walls. For example, Jupiter. The spacecraft approaches Jupiter moving opposite to its orbital motion (Figure 2). In the rest frame of Jupiter, the probe approaches, swings around, and then leaves in (almost) the opposite direction. In the frame of the solar system, the probe gets a boost from Jupiter. This boost is a gravity slingshot. *Puzzle:* Where does the extra energy for the spacecraft come from?

Figure 2. Bouncing off Jupiter. In the lab frame, a spacecraft approaches Jupiter with speed v, as shown in (a). Meanwhile Jupiter is doing its usual orbital motion, giving it speed v_J in the opposite direction. In Jupiter's frame, shown in (b), the spacecraft approaches with speed $v + v_J$, so it bounces off with the same speed. Transforming back to the lab frame shows that the spacecraft leaves Jupiter with speed $v + 2v_J$.

When molecules bounces off a wall, the forces are electromagnetic. When a spacecraft bounces off Jupiter, the force is gravity. But conservation of energy and momentum apply no matter what the forces, so the two systems behave similarly. When a molecule bounces off an approaching wall, it picks up energy (which is how an adiabatic compression raises the temperature of a gas). When a spacecraft 'bounces' off an approaching Jupiter, it too picks up energy.