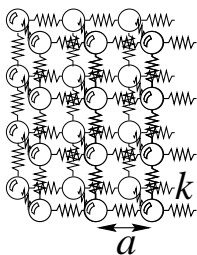# Elasticity − Q4

Model steel.

Example of estimating k given $a = 3 \times 10^{-10}$ m: model the interatomic potential by a quadratic function with minimum at spacing a, and depth 5eV, and with curvature such that the potential is zero when the displacement is $a/2$, gives $k \simeq 1 \, \mathrm{eV}/10^{-20} \, \mathrm{m}^2 = 16 \, \mathrm{N/m}$.

Relate k and a to the Young's modulus, and deduce the Young's modulus of this model steel.

Consider extension e of a tiny cubical (a^3) fragment of steel containing just one bond.

$$Y = (F/A)/(e/l) = (F/e)/a = k/a = 16/3 \times 10^{-10} = 5 \times 10^{10} \, N/m^2$$

[The true value is 2×10^{11} Pa]

We can also estimate Y_{steel} from experience. Imagine plucking a guitar string, and imagine twiddling the peg that tunes the string. From the experience of the forces required to deflect the string sideways 1 cm, or to extend the string by 1 mm, and the resulting change in pitch when this extension is imposed, we can get the information we need. Take the high E string, for example. Its diameter is thin, maybe 0.5 mm. The string is roughly 1 m long, and if we hang a jar of jam (1 kg) from it at its midpoint, it deflects by maybe 1 cm. This gives us the tension, $T = 500$ N (from resolving forces in the long skinny triangle). Sanity check. That means that the tension is about the weight of a 50 kg child. Seems reasonable. Now, how much does extending the wire increase the tension? I'd guess that one revolution of the peg (which extends the string by, say, 1cm) would cause a major change in pitch, maybe as much as a fifth. A fifth is $3/2$ in frequency, which is $9/4$ in tension. So an extension of 1% is expected to double the tension from the starting value. The Young's modulus is the stress that would double the length (*i.e.*, produce a strain of 1), so it's 100 times the stress in the string, *i.e.* (with area $= (.5 \, \mathrm{mm})^2$),
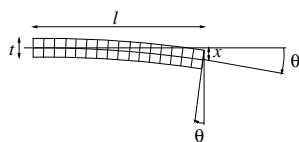
$$Y \simeq 100T/A = 100 \times 500/(25 \times 10^{-8}) = 2 \times 10^{11} \, N/m^2.$$

Lucky!
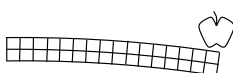
Estimate the vertical deflection of an apple on a ruler. How does your answer depend on the ruler's thickness?

We assume that the upper half of the ruler is stretched and the lower half is compressed. From experience, I'd expect a deflection of 1 mm or 2 cm so.

Define the vertical displacement of the end to be x, the thickness, t, the length l, and the angle of the end of beam, θ. We estimate the energy stored in the ruler when it's deformed as shown. The energy is stored in the stretched and compressed parts. The energy is (typical energy density) $\times$ (volume that is deformed). The angle θ is roughly given by $\theta = x/l$, with a geometry factor of some sort. Shall we take

$$\theta = x/(2l)?$$

The maximum strain, ϵ, of the upper edge is the total extension of the upper edge, $(t/2)\theta$, over l.

$$\epsilon_{\mathrm{max}} = t\theta/(2l)$$

The energy density is

$$\frac{1}{2}Y\epsilon^2.$$

The strain ϵ is a linear function of distance from the midplane, so the energy density is a quadratic function of distance. Let's just use the maximum strain and multiply by half the volume of the beam (assuming that roughly half of it is at the maximum strain). The potential energy is then

$$V(x) \simeq \text{Volume} \times \text{Energy density}$$
$$= \frac{1}{2}(tlw)\frac{1}{2}Y\epsilon^2 = \frac{1}{2}(tlw)\frac{1}{2}Y\frac{t^2(x/(2l))^2}{4l^2} = \frac{1}{2}\frac{t^3wY}{32l^3}x^2,$$

so (comparing this with a Hooke spring's $V = \frac{1}{2}kx^2$) the end of the beam behaves just like Hooke spring with constant

$$k \simeq \frac{t^3wY}{32l^3}.$$

Check dimensions:

$$[F][L]^{-1} \leftrightarrow [L][F][L]^{-2}$$

Notice that this scales as the *cube* of the thickness – thick planks are much harder to bend than thin ones – and it scales inverse-cubically with length, which fits with the experience that an apple deflects a long ruler much more than an equivalent short one. The linear scaling with width w makes complete sense, since two apples on two rulers, side by side, give the same deflection as one apple on one ruler.

So, let's try the apple on our model ruler.

Displacement for a $1\,\text{N}$ apple, on a width-$2\,\text{cm}$ diving board of length $0.3\,\text{m}$, thickness $10^{-3}\,\text{m}$, is predicted to be

$$1\,\text{N}/k \simeq \frac{32l^3}{t^3wY} = \frac{32 \times 0.3^3}{10^{-9} \times .02 \times 2 \times 10^{11}}\,\text{m} = 0.2\,\text{m}.$$

This is an embarrassingly large answer, about ten times larger than expected. The scaling with thickness t is cubic, so big errors arise from getting it a little wrong.

Our estimate of the typical strain for a given deflection is also a possible cause of error, as our estimate of k scales quadratically with the strain.

DJCM. February 7, 2002

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