

SOLUTION TO PROBLEM 3: NORMAL MODES

3.1 A normal mode is a motion in which all particles oscillate at the same frequency. You can find these modes from the equations of motion for the particles. You need one equation per coordinate; the equations usually come from forces and torques. You solve these second-order coupled differential equations, most easily written in matrix form, by assuming an oscillating solution and finding the oscillation frequencies and the normal modes.

3.2 A molecule with N atoms has $3N$ modes, distributed among translation, rotation, and vibration. I often miscount the vibrations if I try to work out the motions from scratch. But there is a shortcut: Count the translations and the rotations and allocate the remainder of the $3N$ to vibrations. Each dimension gives one translation, so there are almost always three translations. A one-atom molecule (for example, helium) has no rotational modes; a linear molecule, such as CO_2 , has two rotational modes; and a nonlinear molecule (such as H_2O) has three rotational modes.

Once I've categorised the modes, I can decide how much each contributes to the specific heat (at constant volume). Each degree of freedom – a quadratic contribution to the internal energy – contributes $R/2$ per mole *if the mode is classical*: if the temperature is high enough. A translational or rotational mode contributes one degree of freedom, from kinetic energy. A vibrational mode contributes two degrees of freedom: from kinetic and potential energies.

The phrase ‘moderately high temperatures’ is distressingly vague. I'll interpret it to mean that the temperature is high enough to make all rotations classical, but not high enough to make every vibration fully classical.

Diatomic hydrogen, H_2 , has six modes: three translations, two rotations, and so one vibration. Assuming that all modes are classical, the specific heat is:

$$C_v = \underbrace{\left(3 \times \frac{R}{2}\right)}_{\text{translation}} + \underbrace{\left(2 \times \frac{R}{2}\right)}_{\text{rotation}} + \underbrace{(1 \times R)}_{\text{vibration}} = \frac{7}{2}R$$

For O_2 , the modes and the specific heat are the same:

$$C_v = \underbrace{\left(3 \times \frac{R}{2}\right)}_{\text{translation}} + \underbrace{\left(2 \times \frac{R}{2}\right)}_{\text{rotation}} + \underbrace{(1 \times R)}_{\text{vibration}} = \frac{7}{2}R.$$

The temperature is probably not high enough to make the H_2 vibration mode classical. The O_2 vibration mode is probably classical because it has a lower level spacing $\hbar\omega$ than H_2 does ($\omega \propto \sqrt{1/m}$).

Water has nine modes: three translations, three rotations, and three vibrations. Its specific heat is

$$C_v = \underbrace{\left(3 \times \frac{R}{2}\right)}_{\text{translation}} + \underbrace{\left(3 \times \frac{R}{2}\right)}_{\text{rotation}} + \underbrace{(3 \times R)}_{\text{vibration}} = 6R.$$

Carbon dioxide also has nine modes, in a different mixture because it is a linear molecule: three translations, two rotations, and four vibrations. Its specific heat is

$$C_v = \underbrace{\left(3 \times \frac{R}{2}\right)}_{\text{translation}} + \underbrace{\left(2 \times \frac{R}{2}\right)}_{\text{rotation}} + \underbrace{(4 \times R)}_{\text{vibration}} = \frac{13}{2}R.$$

Using the ‘moderately high temperature’ to break the tie between H_2 and O_2 , I get this ranking of specific heats:

$$\text{H}_2 < \text{O}_2 < \text{H}_2\text{O} < \text{CO}_2.$$

3.3 (i) I will use any trick to avoid setting up, let alone solving 3×3 matrices; I'll even try the painful process of thinking. So, what motion might be a normal mode?

Guessing normal modes

My first guess is the simplest: All three springs have equal displacement and the plane moves up and down without rocking. Is that motion a normal mode? I push down the plane, keeping it flat, and let go. It bounces up and down, but does it stay flat as bounces? Maybe it rocks as well? Hmm, rocking – that requires a torque. So I compute the torque about the centre-of-mass just as I let go. Fortunately for my lazy self, I don't have to compute it exactly, only check whether it is zero.

The centre-of-mass is along the front bar, one-quarter of the way to the front. Each back spring produces an upwards force F with a lever arm $l/4$, for a torque of $lF/2$. The front spring also produces an upwards force F with a lever arm $3l/4$, for a torque $3lF/4$. Sadly the two torques do not cancel; like Elvis Presley, the plane rocks as it bounces. I must keep looking harder for a normal mode.

A second candidate mode: Jiggle only the front spring, keep the back springs fixed. By now I know to ask: But do the back springs sit quietly as the front dances? Equivalently, does the T-joint try to move when the front tip is struck (with an impulse from the spring)? For this analysis, the back bar does not rock so I can ignore its moment of inertia and collapse it into a point mass at the T-joint.

So I'm holding a cricket bat of mass m and length l , with a blob of chewing gum of mass m stuck at the near end ($x = 0$). A ball moving perpendicular to the bat hits the far end ($x = l$). Does my hand (at the near end) feel a force? Let's say the ball delivers an impulse J , which makes the centre-of-mass (at $l/4$) move with velocity J/m . The ball also makes the rod rotate about the centre-of-mass. It provides an angular impulse L :

$$L = \text{angular impulse} = J \times \text{lever arm} = \frac{3lJ}{4}.$$

This impulse produces an angular velocity ω :

$$\omega = \frac{\text{angular momentum}}{\text{moment of inertia about } l/4}.$$

The moment of inertia is a sum of two moments: from the bat and from the chewing gum. The chewing gum contributes

$$I_{\text{gum}} = m \left(\frac{l}{4} \right)^2.$$

The bat contributes

$$I_{\text{bat}} = \frac{1}{12}ml^2 + \underbrace{m \left(\frac{l}{4} \right)^2}_{\parallel\text{-axis theorem}} = \frac{7}{48}ml^2.$$

The total moment is

$$I = I_{\text{gum}} + I_{\text{bat}} = \frac{5}{24}ml^2.$$

So the angular velocity is

$$\omega = \frac{3lJ/4}{5ml^2/24} = \frac{18}{5} \frac{J}{ml}.$$

If I weren't holding the near end, its velocity would be a combination of the centre-of-mass velocity, moving it forwards, and of the angular velocity scaled by the lever arm ($l/4$), moving it backwards:

$$v_{\text{end}} = \underbrace{\frac{J}{m}}_{\text{linear}} - \underbrace{\frac{9}{10} \frac{J}{m}}_{\text{angular}} = \frac{1}{10} \frac{J}{m}$$

Bad luck. The end wants to move. So the front tip cannot dance without making the back bar dance as well.

Now for the third candidate for a normal mode: The back bar rocks, rotating about the front bar; the T-joint stays fixed. Twisting the back bar, with one spring going up and the other going down by the same amount, creates no net force on the plane, so the centre-of-mass stays put. It also produces no torque trying to tilt the plane forwards (or backwards). So the tip stays put while the back bar twists back and forth. This candidate is a normal mode!

I could lie and say that I thought of this mode using the following symmetry argument. The system has an axis of symmetry: the front rod. So one normal mode is rotation about that axis. In reality, I realised the symmetry argument only after I was sure that rocking was a normal mode. In other words, my hindsight is 20/20, but from now on I'll first look for symmetries.

Finding an oscillation frequency

Now I'll calculate the frequency of the mode and check whether it's one of the three frequencies we're given. For this motion, the front bar becomes a blob of chewing gum at the T-joint. This mode involves rotation, so I solve for the frequency using torques. Imagine twisting the back rod by an angle θ . Each spring is displaced by $\theta l/2$; they provide cancelling forces, each with magnitude $k\theta l/2$, but colluding torques, each equal to $-k\theta(l/2)(l/2)$. The total torque is $-k\theta l^2/2$. The moment of inertia of the back rod with chewing gum is that of the rod: $I = ml^2/12$. The chewing gum has no lever arm, so it contributes no moment. The equation of motion is

$$\text{torque} = \text{moment of inertia} \times \text{angular acceleration}$$

or

$$\frac{1}{2}k\theta l^2 = \frac{1}{12}ml^2 \times \ddot{\theta}.$$

In standard simple-harmonic-motion form,

$$\ddot{\theta} = -\frac{6k}{m}\theta.$$

The oscillation frequency is given by

$$\omega_0^2 = \frac{6k}{m} = \frac{12k}{M}$$

since each rod is half the plane.

Finding the other mode frequencies

The other two modes are trickier to find and to analyse. (How do I know that there are three modes?) For these two modes, I can forget about the back rod twisting – that motion is taken care of by the mode I just calculated. If the back rod doesn't rotate, I can replace it by a piece of chewing gum at the T-joint. Oh happy news, the plane becomes the cricket bat that I analysed before. So all that thinking does not go to waste. Now I also have to include the two back springs (they weren't relevant before, when I was computing the effect of an impulse on the cricket bat). These springs collapse to a single spring at the T-joint, with spring constant $2k$.

The rod has two degrees of freedom, which can be described using various coordinate systems: as displacements of the front and back springs, or as centre-of-mass motion and a rotation (or many others). I can find the normal modes by using the energy method or by using forces and torques. I'll use forces and torques, but you'll learn some physics by doing

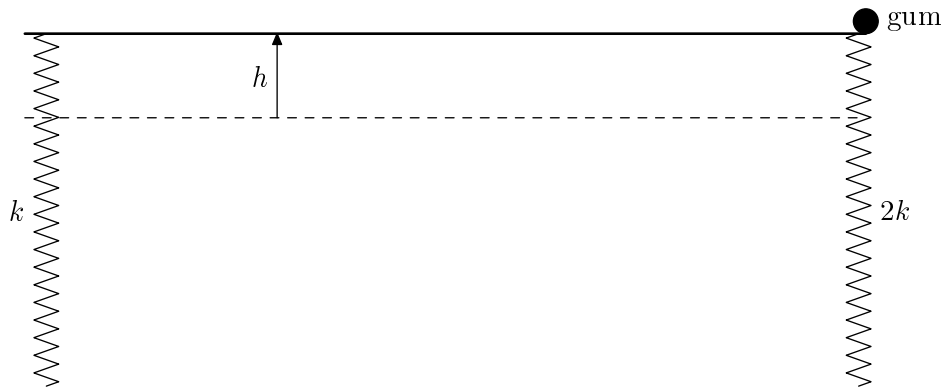


Figure 1. Raising front rod by h .

energy method on your own. To solve for the motion of two coordinates, I need two equations of motion; one comes from forces, the other from torques.

Forces and torques work most naturally with centre-of-mass height (h) and rotation (θ) as the coordinates. I first work out the net force as a function of h and θ . Moving the bat upwards by h (Figure 1) extends each spring by h , giving a total force of $-3kh$; the minus sign means that the springs force the rod downwards. A rotation θ (Figure 2) extends the spring at the chewing-gum end by $\theta l/4$, giving a force of $-2k\theta l/2$. It compresses the other spring (I have to be careful with signs!) by $\theta(3l/4)$, giving a force $3k\theta l/4$. The net force from rotation is $k\theta l/4$.

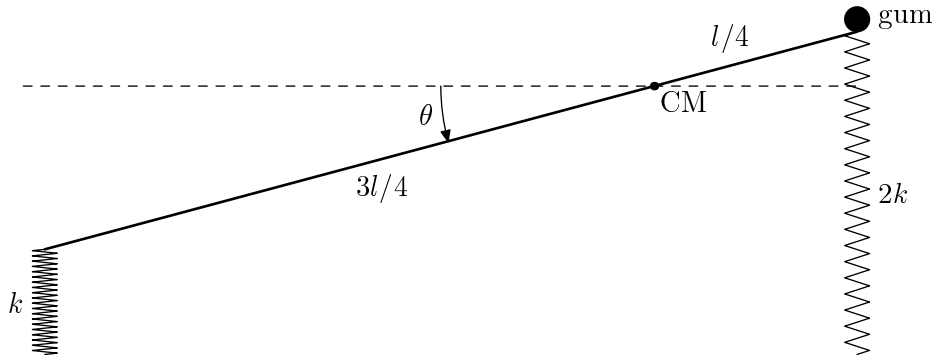


Figure 2. Twisting front rod by θ .

The displacements contributed by the h and θ motions add; the forces therefore add as well:

$$F = -3kh + \frac{kl}{4}\theta.$$

The mass of the system is $2m$, so Newton's second law says that

$$2m\ddot{h} = -3kh + \frac{kl}{4}\theta.$$

For the angular motion, I compute the torques produced by twisting or by moving up and down. If the rod moves up by h , the far spring produces a torque $(3l/4)kh$ and the near spring $-(l/4)2kh$; the sum is $kh l/4$. If the rod twists by θ , the near spring produces a force of $-(l/4)\theta 2k$ and a torque of $-2(l/4)^2\theta k$; the far spring produces a force $(3l/4)\theta k$ and a torque of $-(3l/4)^2\theta k$; the resulting torque is $-11l^2\theta k/16$. Just like the forces, the torques produced by rotation (θ) and by translation (h) add. So the rotational version of Newton's second law says

$$\text{moment of inertia} \times \text{angular acceleration} = \text{torque}$$

or

$$I\ddot{\theta} = \frac{kl}{4}h - \frac{11}{16}l^2k\theta.$$

The moment of inertia, I , has two terms: from the front rod and from the chewing gum. Oh wait, I already calculated I for the intuitive arguments at the beginning:

$$I = \frac{5}{24}ml^2.$$

Pieces from intuitive arguments are often reused in exact calculations; thinking isn't as time-consuming as it seems. Of course, in an exam, curse those stupid things, you may not have time for thinking, as opposed to rapid vomiting of memorised mathematics. But while revising you can develop intuition and understanding.

Now I have equations for \ddot{h} and $\ddot{\theta}$:

$$\begin{aligned}\ddot{h} &= -\frac{3}{2}\frac{k}{m}h + \frac{k}{m}\frac{l}{8}\theta, \\ \ddot{\theta} &= \frac{6}{5l}\frac{k}{m}h - \frac{33}{10}\frac{k}{m}\theta.\end{aligned}$$

Horror. These differential equations are coupled: A twist produces translation, and a translation produces twist. Therefore neither pure translation nor pure twist is a normal mode. Which is what I found in the intuitive arguments. Finding the normal modes means uncoupling the equations: finding two combinations of twist and translation that do not leak into each other.

I try the cheap method first: Stare at the equations and hope that a simple combination pops into mind. But no enlightenment dawns and the messy fractions suggest that none will. So I turn to the official method, writing the two equations as a matrix equation (noting that every term contains a k/m):

$$\frac{d^2}{dt^2} \begin{pmatrix} h \\ \theta \end{pmatrix} = -\frac{k}{m} \begin{pmatrix} \frac{3}{2} & -\frac{l}{8} \\ -\frac{6}{5l} & \frac{33}{10} \end{pmatrix} \begin{pmatrix} h \\ \theta \end{pmatrix}.$$

I factored out k/m and the minus sign to make the equation look like a simple spring equation:

$$\frac{d^2}{dt^2}(\text{blah}) = -\text{something} \times \text{blah}$$

The matrix is a mess because θ and h do not have the same dimensions; the l in the off-diagonal terms sorts out the mismatch. With a new variable, $\psi \equiv l\theta$, the matrix cleans up:

$$\frac{d^2}{dt^2} \begin{pmatrix} h \\ \psi \end{pmatrix} = -\frac{k}{m} \begin{pmatrix} \frac{3}{2} & -\frac{1}{8} \\ -\frac{6}{5} & \frac{33}{10} \end{pmatrix} \begin{pmatrix} h \\ \psi \end{pmatrix}.$$

The problem sheet gives the answer in terms of M , not m ; the matrix equation using M is

$$\frac{d^2}{dt^2} \begin{pmatrix} h \\ \psi \end{pmatrix} = -\frac{k}{M} \begin{pmatrix} 3 & -\frac{1}{4} \\ -\frac{12}{5} & \frac{33}{5} \end{pmatrix} \begin{pmatrix} h \\ \psi \end{pmatrix}.$$

For just the right recombinations of h and ψ , the matrix becomes diagonal:

$$\frac{d^2}{dt^2} \begin{pmatrix} \text{combo 1} \\ \text{combo 2} \end{pmatrix} = - \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \begin{pmatrix} \text{combo 1} \\ \text{combo 2} \end{pmatrix},$$

where ω_1 and ω_2 are the frequencies of the normal modes. Changing to this magic coordinate system doesn't change the trace or determinant of the matrix. So I'll check that the trace and determinant match those in the problem statement.

The k/M factor has the right dimensions: frequency squared. I'm only worried whether the dimensionless constants are right. The matrix

$$\begin{pmatrix} 3 & -\frac{1}{4} \\ -\frac{12}{5} & \frac{33}{5} \end{pmatrix}$$

has trace $48/5$ and determinant $96/5$. The problem claims that apart from the factor of k/M

$$\omega^2 = \frac{24}{5} \left(1 \pm \frac{1}{\sqrt{6}} \right),$$

or that

$$\frac{d^2}{dt^2} \begin{pmatrix} \text{combo 1} \\ \text{combo 2} \end{pmatrix} = -\frac{k}{M} \begin{pmatrix} \frac{24}{5} \left(1 + \frac{1}{\sqrt{6}} \right) & 0 \\ 0 & \frac{24}{5} \left(1 - \frac{1}{\sqrt{6}} \right) \end{pmatrix} \begin{pmatrix} \text{combo 1} \\ \text{combo 2} \end{pmatrix}.$$

The trace is (forgetting about the k/M) $48/5$, which is good so far. The determinant is

$$\frac{24}{5} \left(1 + \frac{1}{\sqrt{6}} \right) \times \frac{24}{5} \left(1 - \frac{1}{\sqrt{6}} \right) = \frac{24 \times 24}{5 \times 5} \left(1 - \frac{1}{6} \right) = \frac{96}{5}.$$

The determinant is also correct. So I've shown that the frequencies of the three normal modes are given by

$$\frac{\omega^2}{k/M} = 12, \quad \frac{24}{5} \left(1 \pm \frac{1}{\sqrt{6}} \right).$$

(ii) What happens if the front wheel goes over a bump? The front wheel gives an upwards kick to the tip of the front rod, making it move up and down. That motion is not a normal mode: it is a combination of the two modes of the front rod, each oscillating at its own frequency. By symmetry, the back rod does not move. So this kick excites just the two front-rod modes. The plane tilts forwards and backwards and moves up and down, but does not rock side to side.

When one of the back wheels goes over the bump, the symmetry argument no longer works; the back rod also rocks and translates. The translation, which moves the T join, has the same effect as translating the front wheel: It excites the two modes of the front rod. So this kick excites all three modes, each oscillating at its own frequency.