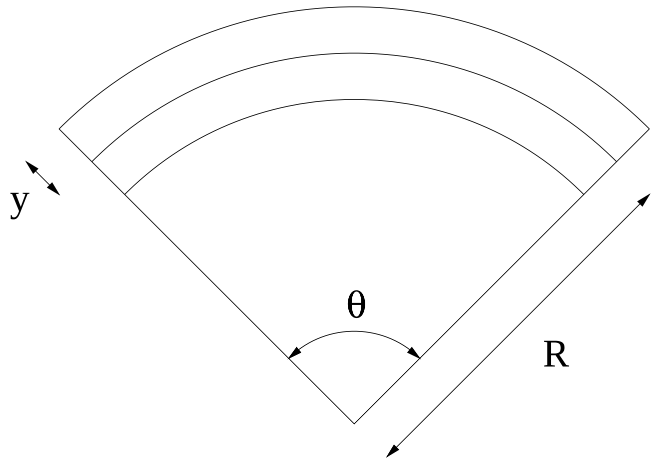
4.1



Since the length of the rod is $l = R\theta$, the strain in the material at y is

$$\epsilon = \frac{(R+y)\theta - R\theta}{R\theta} \tag{1}$$

$$= \frac{y}{R}. \tag{2}$$

The energy density is

$$dE = \frac{Yy^2}{2R^2} dV \tag{3}$$

so the total stored energy in a beam is

$$E = \frac{YIl}{2R^2} \tag{4}$$

$$= \frac{YI\theta^2}{2l} \tag{5}$$

where I is the moment of area

$$I = \int y^2 dA. \tag{6}$$

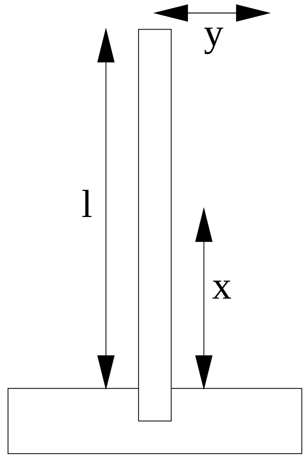
Differentiating we can obtain the bending moment

$$B = \frac{dE}{d\theta} \tag{7}$$

$$= \frac{YI\theta}{l} \tag{8}$$

$$= \frac{YI}{R}. \tag{9}$$

4.2



The bending moment at x due to a force at the end of the ruler is

$$B = F(l - x) \tag{10}$$

we also know that the beam obeys

$$\begin{aligned} B &= \frac{YI}{R} \quad (11) \\ &\approx YI\frac{d^2y}{dx^2} \quad (12) \end{aligned}$$

where I is the moment of area

$$\begin{aligned} I &= \int y^2 dA \quad (13) \\ &= \frac{ab^3}{12}. \quad (14) \end{aligned}$$

Hence

$$\begin{aligned} YI\frac{d^2y}{dx^2} &= F(l - x) \quad (15) \\ YIy &= \frac{F}{6}(l - x)^3 + Ax + B \quad (16) \end{aligned}$$

where A and B are integration constants which can be fixed using the boundary conditions that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$ (since the rod is clamped vertically). Therefore

$$y = \frac{2F}{Yab^3}x^2(3l - x). \tag{17}$$

4.3

The general form for the bending moment is

$$B = \int_{x}^{l} \left(x' - x\right) f\left(x'\right) dx' \tag{18}$$

where $f(x)$ is the force density. Therefore

$$YI \frac{d^2 y}{dx^2} = \int_{x}^{l} \left(x' - x\right) f\left(x'\right) dx'. \tag{19}$$

We can differentiate using the results

$$\frac{dB}{dx} = -\left(x - x\right) f\left(x\right) - \int_{x}^{l} f\left(x'\right) dx' \tag{20}$$

$$= -\int_{x}^{l} f\left(x'\right) dx' \tag{21}$$

$$\frac{d^2 B}{dx^2} = f\left(x\right) \tag{22}$$

to obtain

$$YI \frac{d^4 y}{dx^4} = f\left(x\right). \tag{23}$$

We have the boundary conditions

- $y = \frac{dy}{dx} = 0$ at $x = 0$ since the beam is clamped.
- $\frac{d^2 y}{dx^2} = \frac{d^3 y}{dx^3} = 0$ from $B = \frac{dB}{dx} = 0$ (from equation 18)

$f(x)$ is the force required to hold the beam static. When the beam is unsupported, f is the force that accelerates the beam

$$\frac{d^2 y}{dt^2} dm = -f\left(x\right) dx \tag{24}$$

$$\frac{d^2 y}{dt^2} = -Yab^3 12 \frac{d^4 y}{dx^4} \frac{dx}{dm} \tag{25}$$

$$= -\frac{Yb^2}{12\rho} \frac{d^4 y}{dx^4} \tag{26}$$

where the density is ρ.

When solving, try a solution of the form

$$y = y_0 \exp\left(i\left(kx - \omega t\right)\right) \tag{27}$$

and therefore

$$-\omega^2 + \frac{Yb^2}{12\rho}k^4 = 0. \tag{28}$$

Solutions are of the form $k = \pm q$ and $k = \pm iq$ where

$$q = \sqrt{\omega\sqrt{\frac{12\rho}{Yb^2}}}. \tag{29}$$

Hence a general form for the solution is

$$y = A\cos qx + B\sin qx + C\cosh qx + D\sinh qx. \tag{30}$$

Applying the boundary conditions

- $y = 0$ at $x = 0$ so $A + C = 0$
- $\frac{dy}{dx} = 0$ at $x = 0$ so $B + D = 0$
- $\frac{d^2y}{dx^2} = 0$ at $x = l$ so $-A\cos ql - B\sin ql - A\cosh ql - B\sinh ql = 0$
- $\frac{d^3y}{dx^3} = 0$ at $x = l$ so $A\sin ql - B\cos ql - A\sinh ql - B\cosh ql = 0$

Therefore

$$\begin{aligned} A\left(\cos ql + \cosh ql\right) &= -B\left(\sin ql + \sinh ql\right) && (31)\\ B\left(\cos ql + \cosh ql\right) &= A\left(\sin ql - \sinh ql\right) && (32)\end{aligned}$$

and so

$$\begin{aligned} \left(\cos ql + \cosh ql\right)^2 &= -\left(\sin^2 ql - \sinh^2 ql\right) && (33)\\ \cos^2 ql + \sin^2 ql + \cosh^2 ql - \sinh^2 ql + 2\cos ql\cosh ql &= 0 && (34)\\ \cos ql\cosh ql &= -1 && (35)\end{aligned}$$

So if we define α to be a solution of $\cos\alpha\cosh\alpha = -1$ then

$$\begin{aligned} q &= \frac{\alpha}{l} && (36)\\ \omega &= \frac{\alpha^2 b}{2l^2}\sqrt{\frac{Y}{3\rho}} && (37)\end{aligned}$$