

# 1B Dynamics Solutions

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February 2, 2000

Most of the content of this document derives from Dr David J C MacKay's on-line scans; except notably answers to **T.26, T.28, T.29, T.30, T.31, T.32.**

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## I. LECTURE EXAMPLES

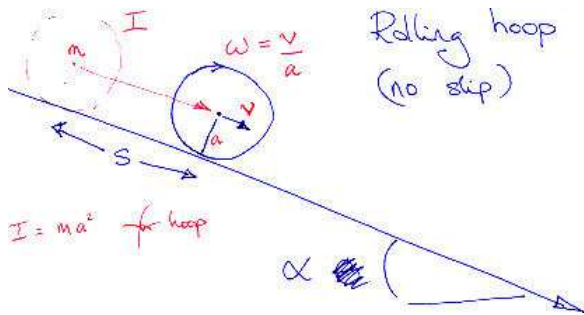
**L.1 Moment of Inertia.** Moment of inertia are in the following order:

ring > shell > disk > sphere

This can be derived by considering  $\sum r^2$  or the  $\langle r^2 \rangle$  for each of them; i.e. how far the average mass is away from the axis.

When rolling down the hill, gravity has to drive both the rotation and the linear motion. So for the same mass, objects with higher moment of inertia roll slower.

Hoop :  $I = mr^2$ , disk :  $I = \frac{mr^2}{2}$ . Hoop slower.



$$\begin{aligned}
 E &= T + V \\
 &= \text{rotation k.e.} + \text{k.e. of c.g.} + mgs \sin \alpha \\
 &= \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 + mgs \sin \alpha \\
 &= \frac{1}{2} I \frac{\dot{s}^2}{a^2} + \frac{1}{2} m \dot{s}^2 + mgs \sin \alpha \\
 &= \frac{1}{2} \left( \frac{I}{a^2} + m \right) \dot{s}^2 + mgs \sin \alpha
 \end{aligned}$$

$$\begin{aligned}
 \frac{dE}{dt} &= 0 = \left( \frac{I}{a^2} + m \right) \dot{s} \ddot{s} + mg \dot{s} \sin \alpha \\
 \Rightarrow \ddot{s} &= -\frac{g}{1 + \frac{I}{ma^2}} \sin \alpha
 \end{aligned}$$

(Check dimension and check the special case of  $\alpha = 0$  and  $\alpha = \pi/2$ .)

\* Note no need to evaluate reaction force nor friction.

**L.2 Clock** • (1st)  $g \propto r^{-2}$ , so  $\frac{\delta g}{g} = -\frac{2\delta r}{r}$ .  $r_e \approx 6000 \text{ km}$ . Any reasonable number for  $\delta r$  (e.g. a few metres).

• (2nd)  $\tau \propto \sqrt{\frac{l}{g}}$ , so  $\frac{\delta \tau}{\tau} = -\frac{\delta g}{2g} = \frac{\delta r}{r}$ .

• (3rd) To do it exactly,

$$E(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + (1 - \cos \theta) m g l$$

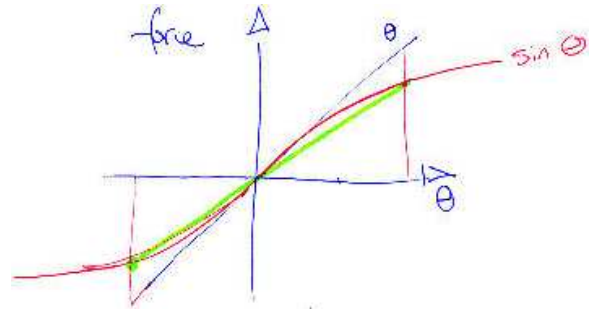
$$\epsilon = \frac{E}{m g l} = \frac{1}{2} \frac{\dot{\theta}^2}{\omega_0^2} + (1 - \cos \theta)$$

Rearranging, and using the fact that  $\epsilon = 1 - \cos \theta_{\max}$  we eventually get:

$$\frac{1}{\omega_0} \int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{2(\cos \theta - \cos \theta_{\max})}} = \int_0^{T/4} dt$$

Evaluating this integral would get an exact answer.

On the other hand, the difference can be approximated by changing  $g$  to  $g'$  by the same factor as  $\langle \theta \rangle \sim \langle \sin \theta \rangle$ .  $\tau \propto g^{-1/2}$ .



$$\sin \theta = \theta - \frac{\theta^3}{6} + \dots,$$

$$\text{force} \approx \frac{\theta}{\theta_{\text{typical}}} \left\{ \theta_{\text{typical}} - \frac{\theta_{\text{typical}}^3}{6} + \dots \right\},$$

$$\text{i.e. } \frac{k_{\text{typical}}}{k} = 1 - \frac{\theta_{\text{typical}}^2}{6} + \dots,$$

Since  $\omega^2 \sim k$ , therefore

$$\frac{\delta(\omega^2)}{(\omega^2)} = \frac{\delta k}{k} = -\frac{\theta_{\text{typical}}^2}{6} + \dots$$

Period  $T \sim \frac{1}{\omega}$ , i.e.

$$\frac{\delta T}{T} = -\frac{1}{2} \cdot \frac{\delta(\omega^2)}{(\omega^2)} = \frac{\theta_{\text{typical}}^2}{12} + \dots$$

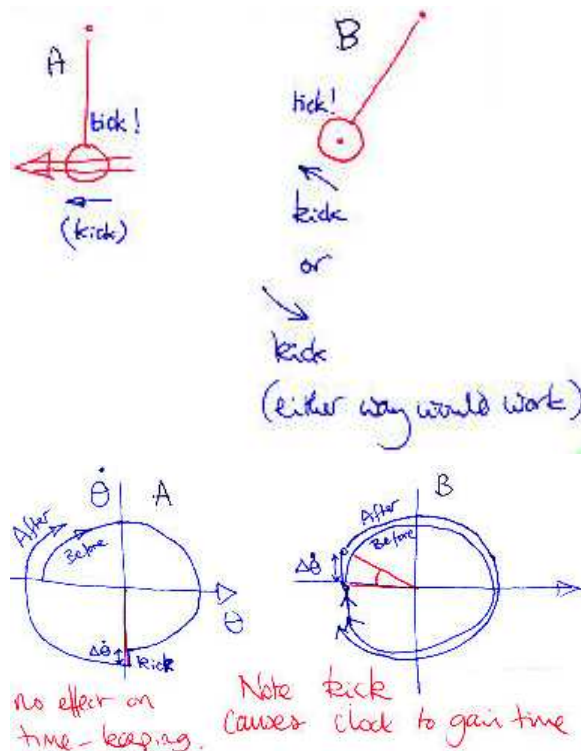
Let  $\theta = \frac{10}{360} \cdot 2\pi \approx 0.17 \implies \frac{\delta T}{T} \approx 0.0025$

That's 220 s per day, i.e. three and half minutes per day.

Conclusion: For time-keeping, either correct for non-linearity or keep amplitude very small.

• (4th) Kick at the lowest point. Kicking at the top would put the state of the pendulum into a different point in the cycle (skipping forward or backward a fraction of a period). Therefore the clock would gain or lose time gradually. E.g. kicking at the extreme point puts it into a state such that it can go further before it reaches the extreme, therefore this short travel time from the old extreme point to the new one is a time lapse every time it is kicked.

State space diagram:



• (5th) Fractional time change per day equals to the fractional error in longitude per  $360^\circ$ , which in turn equals to fractional change to circumference of the earth ( $2\pi \times 6000$  km).

**L.3 Slinky** Only dimensionless group is  $\frac{\tau^2 k}{m}$ . Therefore:  
 $\tau \propto \sqrt{\frac{m}{k}}$ .

**L.4 Triangle** (1)  $A = a^2 F\left(\frac{a}{b}\right)$ , in fact  $A = \frac{ab}{2}$ .  $F(x) = \frac{1}{2x}$ .  
 (2)  $c^2 = a^2 F\left(\frac{a}{b}\right)$ ,  $F(x) = 1 + \frac{1}{x^2}$ .

**L.5 Inverse-square orbit.** One notable point: kicking radially has no effect on the angular momentum, which fixes the semi-latus-rectum (half the width at the position of the focus). So the new orbit must pass through the same two diametrical points from the centre.

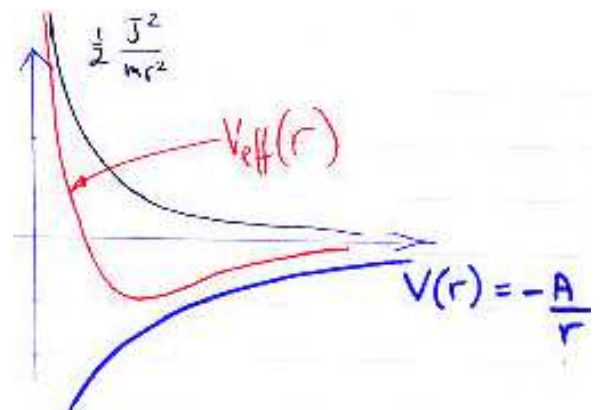
$$E = \text{k. e.} + \text{p. e.} \\ = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{A}{r}$$

$$J = m r^2 \dot{\theta}$$

Therefore,

$$E = \frac{1}{2} m \dot{r}^2 + \left[ \frac{J^2}{2 m r^2} - \frac{A}{r} \right] \\ = \text{radial effective k. e. } T_{\text{eff}}(\dot{r}) \\ + \text{effective potential } V_{\text{eff}}(r)$$

if  $\left. \frac{\partial V_{\text{eff}}}{\partial r} \right|_{r_0} = 0$ , then we get  $\frac{J^2}{m r_0^3} = \frac{A}{r_0^2}$ , the radius  $r = r_0$  for circular motion at constant rate.

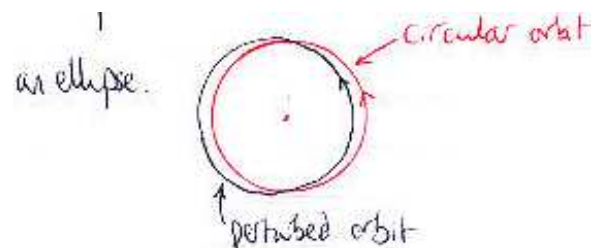


To obtain the frequency of small oscillation of  $r$  around  $r_0$ , evaluate the 2nd derivative:

$$\left. \frac{\partial^2 V_{\text{eff}}}{\partial r^2} \right|_{r_0} = \frac{A}{r_0^3} = \frac{J^2}{m r_0^4}$$

And  $\omega_{\text{SHM}}^2 = \frac{1}{m} \frac{\partial^2 V_{\text{eff}}}{\partial r^2} = \dot{\theta}^2$ . i.e. SHM in  $r$  (in-out motion) at the same rate as rotation.

Coincidence: Perturbed orbit is also a closed orbit (in fact the familiar ellipse).



The case  $V(r) = -\frac{A}{r^{1+\alpha}}$ :

$$\left. \frac{\partial V_{eff}}{\partial r} \right|_{r_0} = 0 \text{ gives } \frac{J^2}{mr_0^3} = (1+\alpha) \frac{A}{r_0^{2+\alpha}},$$

Frequency of small oscillation:

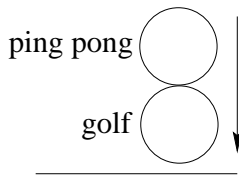
$$\left. \frac{\partial^2 V_{eff}}{\partial r^2} \right|_{r_0} = (1-\alpha^2) \frac{A}{r_0^{3+\alpha}} = (1-\alpha) \frac{J^2}{mr_0^4}$$

$$\text{Thus } \omega_{SHM}^2 = (1-\alpha) \frac{J^2}{m^2 r_0^4} = (1-\alpha) \dot{\theta}^2$$

For  $\alpha > 0$ ,  $\omega_{SHM}$  is a little smaller than  $\dot{\theta}$ , and the oscillation takes slightly longer to intersect the circle than rotation round the centre. Therefore the ellipse precesses in the direction of  $\dot{\theta}$ .

(\* General relativity predicts a tiny correction to the inverse square law; in the vicinity of a massive star the gravitation field is very slightly stronger than inverse square (equivalent to  $\alpha > 0$ ), and the effect is observable in Mercury's orbit as a tiny advance of the perihelion (the near point) per revolution.)

**L.6 Ping-pong** Golf ball goes back as  $-v$ , ping pong ball as  $-3v$ . The motion of the golf ball reverses first, then the two balls are moving at  $-v$  and  $v$  respectively; i.e. relative velocity  $2v$ . At the moment of impact, due to the big difference in mass, the ping pong motion is simply reversed in their centre of mass frame (which is effectively in which the golf ball is at rest), if the collision is elastic.



**L.7 Bead on wire** Let  $\dot{\phi} = \omega$ , the constant rate of rotation around the centre.

$$T = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} (q \sin \theta)^2 \dot{\phi}^2$$

$$V = mg \cos \theta$$

$$\text{momentum: } \frac{\partial L}{\partial \dot{q}} = m \dot{q}$$

$$\text{force: } \frac{d}{dt}(m \dot{q}) = \frac{\partial L}{\partial q} = m q \sin^2 \theta \dot{\phi}^2 - m g \cos \theta$$

The first term on the right is similar to the component of a centrifugal force along the wire. Therefore,

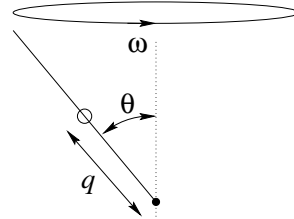
$$m \ddot{q} = m q \sin^2 \theta \dot{\phi}^2 - m g \cos \theta$$

or

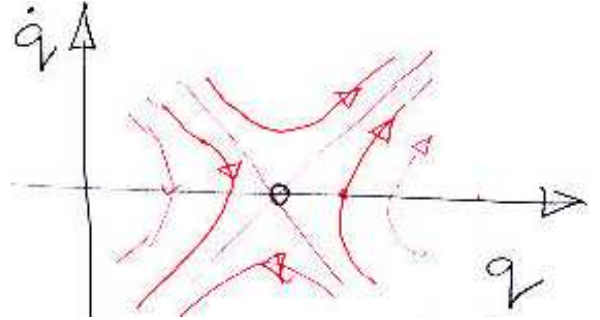
$$m \ddot{q} = A q - B$$

Note that the system energy is not conserved.

There is one fixed point ( $\ddot{q} = 0$ ) at  $q \sin^2 \theta \dot{\phi}^2 = g \cos \theta$ . This is an unstable fixed point  $q_{fp} = \frac{g \cos \theta}{\sin^2 \theta \dot{\phi}^2}$ .



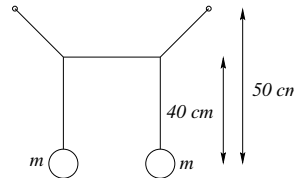
The fixed point is unstable: if the initial  $q < q_{fq}$ , the bead spiral into the bottom; if  $q > q_{fq}$ , the beam will fly out into the air.



**L.8 Coupled pendula.** Total of 5 degrees of freedom - 3 translations x/y/z for each ball, minus one constraint due to unstretchable string in the middle.

em (Hmm, this is rather sloppy; the proper consideration might be to consider rotation around joints, etc. There are 3 translations for each ball, 2 rotations around each of 6 joints = 18 freedoms total; 5 fixed lengths, 2 fixed points from which the whole ensemble hangs, and the fact that the rotations of the middle bar about its two ends must necessarily agree. So we have  $3 \times 2 + 2 \times 6 - 5 - 2 \times 3 - 2 = 5$ )

There are 5 normal modes. Swinging in plane in-phase, out-of-phase, and swinging out of plane in-phase and out-of-phase, and lastly, one going down while the other going up. (These are approximate descriptions only.)



When one bob is displaced a little out of plane and released, the subsequent motion would consist of the superposition of the two out-of-plane modes, and the time taken for energy to travel to and fro is simply the period of the beating frequency.

The frequency of the in-phase motion is  $\omega_1^2 = \frac{g}{l_1}$ , but the frequency of the out-of-phase motion requires slightly more consideration. The mass would be effectively suspended and oscillating with a effective length somewhat between  $l_1$  and  $l_2$  for the geometry shown. Let's say exactly in the middle. Then  $(\omega_2')^2 = \frac{2g}{(l_1+l_2)}$ . In any case,

$$\frac{\Delta\omega}{\omega} \approx -\frac{1}{2} \cdot \frac{\Delta l}{l}$$

Beating frequency  $|\Delta\omega| \approx \frac{1}{2} \cdot \frac{\Delta l}{l} \omega$ , Time taken for energy to travel to and fro is  $2\pi$  over this; and the number of swings to achieve this is  $N = \frac{\omega}{|\Delta\omega|} \approx 2 \cdot \frac{l}{\Delta l}$

The correct answer will require a length in-between  $l_1$  and  $l_2$  whose value depends on the geometry of the strings. For example, if the top string is very long, and the two masses are suspended close together, then the length will be close to  $l_2$ .

**L.9 Slingshot** Let  $R_e$ ,  $R_j$ ,  $R_s$  be the radii of the orbit of Earth, Jupiter, Saturn respectively.

Without slingshot we need to get from Earth to Saturn's orbit directly,

$$\Delta v_1 = \sqrt{\frac{GM}{R_e}} \left( \sqrt{\frac{2}{1 + \frac{R_e}{R_s}}} - 1 \right)$$

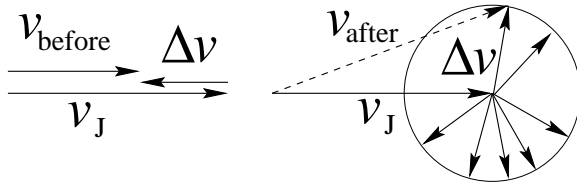
With slingshot, we approximately only need to get to Jupiter's orbit, and the rest is taken care of,

$$\Delta v_2 \approx \sqrt{\frac{GM}{R_e}} \left( \sqrt{\frac{2}{1 + \frac{R_e}{R_j}}} - 1 \right)$$

By symmetry, it would arrive with velocity:

$$\Delta v_3 = \sqrt{\frac{GM}{R_j}} \left( \sqrt{\frac{2}{1 + \frac{R_j}{R_e}}} - 1 \right) \approx -0.4V_j$$

( $R_j \approx 5R_e$ ) Afterwards, the maximum speed it can have is  $|\Delta v_3|$  forward,



and this is greater than required to get to Saturn:

$$\Delta v_4 = \sqrt{\frac{GM}{R_j}} \left( \sqrt{\frac{2}{1 + \frac{R_j}{R_s}}} - 1 \right) \approx 0.15V_j$$

Approximately  $R_j \sim 5R_e$ ,  $R_s \sim 2R_j \sim 10R_e$ , we obtain  $\Delta v_1 = 0.35v_e$ ,  $\Delta v_2 = 0.29v_e$ ,  $\Delta v_3 = -0.42v_j$ , and  $\Delta v_4 = 0.15v_j$ . The energy saving is  $\propto \frac{\Delta v_2}{\Delta v_1}$  (thrust, or fuel usage), i.e. about 16% or about one 6th.

The reason why in actual plans, Venus is often used in the first stage is because getting to Venus requires  $\Delta v_5 = 0.09v_e$  because the Venus-sun distance is quite close to Earth-sun; and its small period means the correct orientation is more frequent (than Mars).

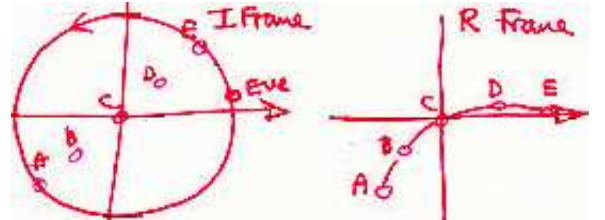
**L.10 Cross Section** for a hard sphere it is simply  $\propto a^2$ , independent of the incident velocity. For inverse-square interactions it would be  $\propto \left( \frac{A^2}{m^2 v^4} \right)$

**L.11 Ride** The occupant is held onto the wall by friction, which is proportional to the normal reaction. The normal reaction is such that to keep the occupant in circular motion, so it has to provide for the necessary centripetal force — or in other words, equal and opposite to the fictitious centrifugal force.

In both cases the flight path of the ball as viewed by the occupant in the rotating frame would be some curved path, only difference is that in (b) the arc will pass through the centre as viewed by the moving occupants.

In the inertial frame, the path when viewed from above is a STRAIGHT LINE.

Also, if the path passes over the centre in one frame, it must pass over the centre in the other frame.



**L.12 Fallen Rulers** The correct answer is this:

In both situations, we can find the final velocities by conservation methods. (Momentum, and Energy) In A, there is no horizontal force so there is zero horizontal momentum at the end.

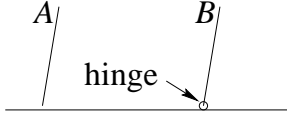
In B, there is a horizontal force, but at the moment when the ruler is horizontal, the horizontal momentum of the ruler must be zero because it is instantaneously rotating about the hinge with a velocity that is vertical.

So if we assume that in A the ruler's left tip is in contact with the surface at all times, the speeds of the tips of A and B at impact will be IDENTICAL.

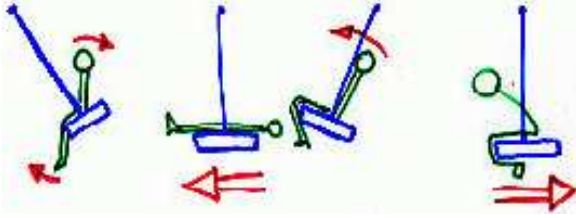
(ed: Magnitudes of speed are the same, in fact the same dependence on the position of the c.g. by energy arguments, but A is moving sideways, so vertical component smaller...)

The times taken for the two rulers to fall will be different.

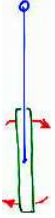
The assumption that ruler A's left tip remains in contact with the surface at all times is questionable, however. Future version of the question (according to Dr Mackay) would be rewritten so that this constraint is enforced, e.g. by attaching ruler A to a heavy trolley.



**L.13 Swing** A swinger keeps going by retracting and extending his legs, mainly. This changes the position of the c.g. (and hence the torque around the suspension point) and pump energy into the swing.

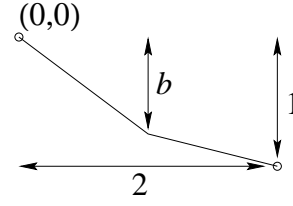


It is possible to start from rest by move the whole body to and fro to start the motion:



(Much more mathematical details in Dr MacKay's scan.)

**L.14 Fastest path** The qualitative characteristic of the path should be such that it is vertical at the beginning to take advantage of gravity to pick up speed as soon as possible so that it travels most of the time at the highest speed; and at the end horizontal so that it covers the horizontal distance quickly. The shape is a cycloid - it is the path described by a point painted on the side of the wheel while it rolls:  $x = A\theta + B \sin \theta + C$  and  $y = D \cos \theta + E$  (note the rolling term  $A\theta$  in  $x$ ).



(2nd) For two straight pieces, meeting at a point  $(a,b) = (1,b)$ . In the first section the acceleration is  $g \frac{b}{\sqrt{1+b^2}} = g \sin \theta_1$  while in the second section the acceleration is  $g \frac{1-b}{\sqrt{1+(1-b)^2}} = g \sin \theta_2$  parallel to the plane. Note that  $\tan \theta_1 + \tan \theta_2 = 1$ .

Consider the vertical motion alone. The component of the accelerations are:  $g_1 = g \frac{b^2}{1+b^2} = g \sin^2 \theta_1$  and  $g_2 = g \frac{(1-b)^2}{1+(1-b)^2} = g \sin^2 \theta_2$  respectively. Just before the junction, the velocity is  $\sqrt{2gb}$ , and its projection parallel to the 2nd part is:  $\sqrt{2gb} \cos(\theta_2 - \theta_1)$  so the vertical component is  $\sqrt{2gb} \cos(\theta_2 - \theta_1) \sin \theta_2$ . The time for the 1st part is given by  $t_1 = \sqrt{\frac{2b}{g_1}} = \sqrt{\frac{2b}{g} \frac{1}{\sin \theta_1}}$ . The time  $t_2$  in the second section satisfies:

$$\begin{aligned} \frac{g_2}{2} t_2^2 + \sqrt{2gb} \cos(\theta_2 - \theta_1) \sin \theta_2 t_2 &= (1 - b) \\ t_2 &= \frac{1}{g_2} \left\{ -\sqrt{2gb} \cos(\theta_2 - \theta_1) \sin \theta_2 \right. \\ &\quad \left. + \sqrt{2gb \cos^2(\theta_2 - \theta_1) \sin^2 \theta_2 + 2g_2(1 - b)} \right\} \\ &= \frac{1}{g \sin^2 \theta_2} \left\{ -\sqrt{2gb} \cos(\theta_2 - \theta_1) \sin \theta_2 \right. \\ &\quad \left. + \sqrt{2gb \cos^2(\theta_2 - \theta_1) \sin^2 \theta_2 + 2g \sin^2 \theta_2(1 - b)} \right\} \\ &= \sqrt{\frac{2b}{g}} \frac{-\cos(\theta_2 - \theta_1) + \sqrt{\cos^2(\theta_2 - \theta_1) + \frac{(1-b)}{b}}}{\sin \theta_2} \end{aligned}$$

$$\begin{aligned} \tau &= t_1 + t_2 \\ &= \sqrt{\frac{2b}{g}} \left( \frac{1}{\sin \theta_1} \right. \\ &\quad \left. + \frac{-\cos(\theta_2 - \theta_1) + \sqrt{\cos^2(\theta_2 - \theta_1) + \frac{(1-b)}{b}}}{\sin \theta_2} \right) \\ &= \sqrt{\frac{2b}{g}} \left( \frac{1}{\sin \theta_1} + \frac{1}{\sin \theta_2} \right. \\ &\quad \left. + \frac{-\cos(\theta_2 - \theta_1) - 1 + \sqrt{\cos^2(\theta_2 - \theta_1) + \frac{\tan \theta_2}{\tan \theta_1}}}{\sin \theta_2} \right) \end{aligned}$$

Hmm, this is a mess. Maybe it is meant to be a mess. The solution is not going to be symmetrical in  $(\theta_1, \theta_2)$  (,or in the  $(b, (1-b))$  pair). Obviously if  $\theta_1 = 0$  it isn't going to move an inch, but if  $\theta_2 = 0$  the decent time is finite.

Dr Mackay: One simply works out  $t$  as a function of  $b$ ; I didn't get to this in lectures, but my plan was to plot the result on a computer. It is interesting that the optimal value of  $b$  is greater than 1, a result that I think few of them would have anticipated.

Maybe  $b > 1$  benefits from the idea that a steeper fall at the beginning picks up a lot of momentum and that's more than offsetting the disadvantage of an awkward (and even upward) bend later.

## II. TRADITIONAL PROBLEMS

### T.1 Pulleys (a)

Geometry:  $z_2 = -2z_1$

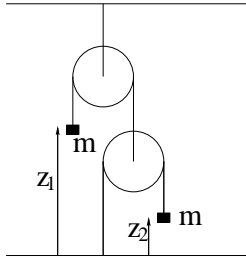
K. E.:  $T = \frac{m}{2}[\dot{z}_1^2 + \dot{z}_2^2]$

P. E.:  $V = mg(z_1 + z_2) = -mgz_1$

Then just do  $E = T + V$  and  $\frac{dE}{dt} = 0$  gives,

Ans:  $\ddot{z}_1 = +\frac{g}{5}$ ,  $\ddot{z}_2 = -\frac{2g}{5}$

[Ans:  $\frac{1}{5}g$ ,  $-\frac{2}{5}g$ ]



(b) This is actually quite tricky - on first sight everything seems to be slower due to added rotational inertia, but one of the pulleys is suspended in air and this affects the whole dynamics. In fact the actual accelerations depends on the ratio of the masses and the pulleys; and the result can be either.

Geometry:  $z_2 = -2z_1$

P. E.:  $V = mg(z_1 + z_2) - Mgz_1 = -(m + M)gz_1$

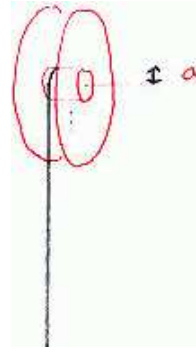
K. E.:  $T = \frac{1}{2}[(m + M)\dot{z}_1^2 + m\dot{z}_2^2 + 2I\omega^2]$

where  $\omega = \frac{\dot{z}_1}{a}$  (note both pulleys have same angular velocity).

So using  $E = T + V = \frac{1}{2}[m + M + 4m + \frac{2I}{a^2}]\dot{z}_1^2 - (m + M)gz_1$ ,  $\frac{dE}{dt} = 0$  gives:

$$\ddot{z}_1 = \frac{(m + M)g}{5m + M + \frac{2I}{a^2}}$$

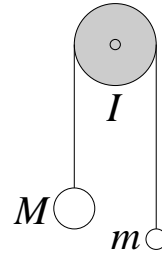
Compare this with  $\ddot{z}_1 = +\frac{g}{5}$  in (a), could be bigger or smaller. Let  $I = \alpha Ma^2$  (e.g. for a hollow loop,  $\alpha = 1$ ), then for  $(1 + 2\alpha) < 5$ , with massive pulleys is faster than with massless pulleys. But with cumbersome pulleys like this ( $\alpha \gg 1$ ):



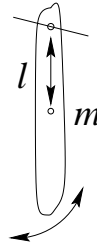
The acceleration will be slower. It'll also be slower if the upper pulley has much bigger mass than the lower one.

$$(c) E = \frac{1}{2}(M + m)\dot{z}^2 + \frac{1}{2}\frac{I}{a^2}\dot{z}^2 + (m - M)gz$$

$$\ddot{z} = \frac{(M - m)g}{(M + m + \frac{I}{a^2})}$$



### T.2 Compound pendulum (a)



$$I = I_0 + ml^2 = m(k^2 + l^2),$$

$$E = T + V = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mgl\theta^2,$$

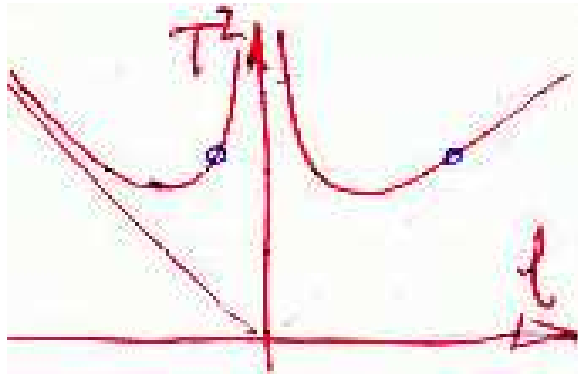
$$m(l^2 + k^2)\ddot{\theta} = mgl \sin \theta$$

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}$$

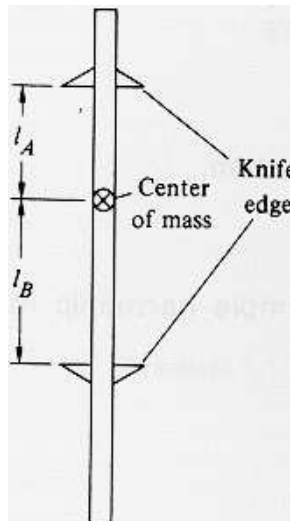
$$\text{Period: } \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

At small  $l$ , rotational inertia dominated, at large  $l$ , translational inertia dominated; in the middle, minimum inertia (smallest period).

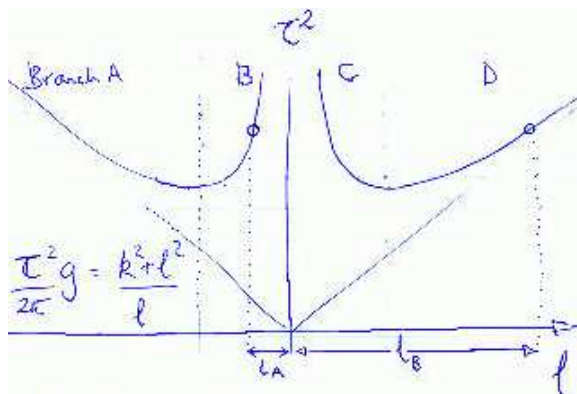




(b,c)

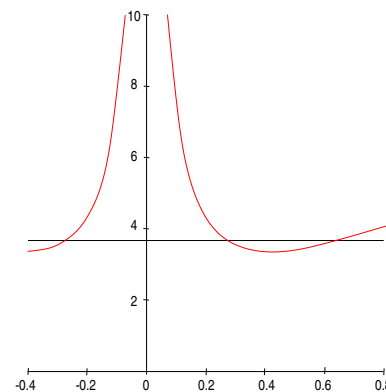


In Kater's pendulum, a pair of suspension points are found such that they have the same  $\tau^2$ , and are on different branches B and D as shown.



The problem is that, the  $(l_A + l_B)$  dependence on  $T$  is actually quite weak and insensitive for values of  $l_A \approx l_B$ , and it is quite possible to mistakenly use the wrong branch. So to get accurate and unambiguous readings, since there are actually 4 points having the same period, you have to choose  $l_A \gg l_B$  or vice versa to get the correct pair. These have a rather much larger  $\tau = \tau_A = \tau_B$  which is going to be difficult to measure; because traditionally to measure the period to a certain accuracy we just

time a certain number of periods so the longer the actual period, the less convenient it is to be measured to a required accuracy. On the other hand, The matching condition of  $\tau_A = \tau_B$  may not be convenient (requiring holes densely populated enough on the beam to attach the knife edges) nor accurate



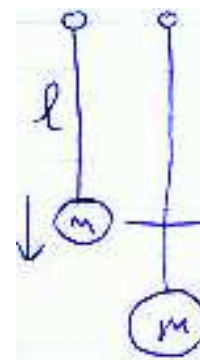
the most accurate measurement will be made if  $\tau^2$  is sensitive to  $l$ , and  $l_A + l_B$  is as big as possible.

better choice

bad choice because  $\tau^2$  is a flatter function of  $l$ .

**T.3 Safety rope** Using dimensional analysis, we obtain  $F \propto mg f(\frac{mg}{k^*})$ , no dependence on  $l$ .

Energy conservation:  $\frac{1}{2}kx^2 - mg(x+l) = 0$ , equalization at both end points (same energy and both velocity zero). This is a quadratic in  $x$ .



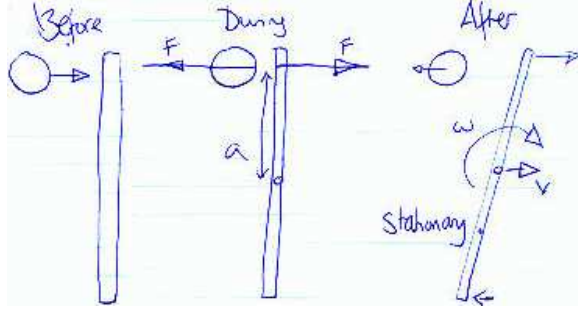
The rope would stretch until

$$\frac{k^* x^2}{2l} = m(l+x)g$$

where  $x$  is the extension. When (a) stiff,  $x \ll l$ , ignore 1st term in right,  $F = kx = \sqrt{2k^*mg}$ , while (b) very stretchable,  $x \gg l$ , ignore 2nd term on left,  $F = 2mg$ .

Note in the 2nd case, the restoring force is independent of the spring's characteristics or its length.

**T.4 Ball games** A transverse momentum impulse hitting one end of a freely suspended bar; the impulse makes the bar fly off with constant velocity and also starts rotating around its c. g. The combined motion is such that the end being hit flies forward at a certain speed, while the other end swings backward at half that speed in the lab frame, so that the point (2/3) from the end being hit is instantaneously stationary, and should be where somebody holds it to minimize the pain.



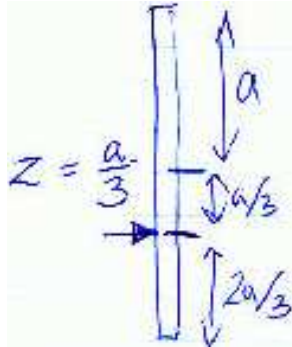
Angular momentum :  $I\omega = Pa$ ,

Momentum:  $v = \frac{P}{m}$ ,

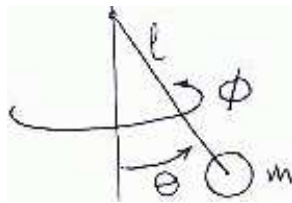
stationary point is given by:

$v = \omega z$ , i.e.

$$z = \frac{v}{\omega} = \frac{I}{ma} = \frac{a}{3}$$



## T.5 Conical pendulum



$$T = \frac{m}{2} \left[ (l\dot{\theta})^2 + (l\dot{\phi} \sin \theta)^2 \right]$$

$$V = mgl(1 - \cos \theta)$$

$$E = T + V = V_{\text{eff}} + \frac{1}{2}ml^2\dot{\theta}^2$$

$$\text{where } V_{\text{eff}} = V + \frac{1}{2} \frac{J^2}{ml^2 \sin^2 \theta}$$

(a) Moving steady in a circle means stationary in the effective potential, therefore  $\theta = \theta_0$  is such

that  $\left. \frac{\partial V_{\text{eff}}}{\partial \theta} \right|_{\theta_0} = mgl \sin \theta - \frac{J^2 \cos \theta}{ml^2 \sin^3 \theta} = 0$ ; Putting the angular momentum about the center:  $J = (ml^2 \sin^2 \theta) \dot{\phi}$  back into this equilibrium condition and using the fact that rotating angular velocity:  $\Omega = \dot{\phi}$ , the result follows.

(b) oscillation about  $\theta_0$  is related to the 2nd derivative  $\left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta_0}$  of  $V(\theta)$  about  $\theta_0$ . Rewrite the energy as a Taylor series:

$$E = V(\theta_0) + \frac{1}{2!} \left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta_0} (\Delta\theta)^2 + \dots + \frac{1}{2} ml^2 \dot{\theta}^2$$

The oscillation frequency  $\omega$  is given by:

$$\omega^2 = \frac{1}{ml^2} \left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta_0}$$

(c.f.  $E = E_0 + \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$  for SHM at frequency  $\omega^2 = k/m$ )

$$\frac{\partial^2 V}{\partial \theta^2} = mgl \cos \theta + \frac{J^2}{ml^2} \frac{2 \cos^2 \theta + 1}{\sin^4 \theta}$$

Simplifying, we get

$$\omega^2 = \frac{1}{ml^2} \left. \frac{\partial^2 V}{\partial \theta^2} \right|_{\theta_0} = \frac{g}{l \cos \theta} (1 + 3 \cos^2 \theta)$$

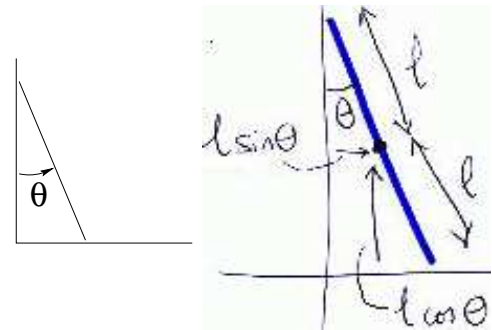
When  $\theta_0 \approx 0$ ,  $\cos \theta \approx 1 \rightarrow \omega \approx 2\Omega$ , i.e. a central ellipse. This is hardly surprising, because the linear pendulum (and superpositioning two orthogonal ones) is certainly a viable motion.

On the otherhand, expanding for small  $\theta_0$ ,

$$\frac{\omega^2}{4\Omega^2} = 1 - \frac{3\theta_0^2}{8} + \dots$$

so the small oscillation will not quite have finished in one big revolution. Therefore the precession is such that the orientation of the ellipse will precess in the direction of  $\theta$ .

## T.6 Ladder



For length = 2l:

$T = \text{rotation about c.g.} + \text{linear motion of c.g.} = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$



$$(\dot{x}^2 + \dot{y}^2) = l^2 \dot{\theta}^2,$$

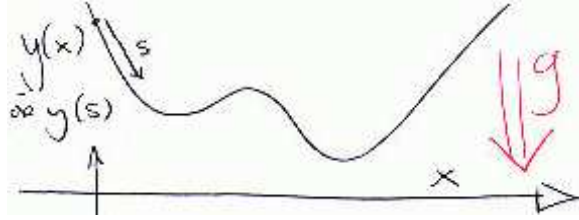
$$V = -mgl(1 - \cos \theta),$$

Equation of motion:

$$(I + ml^2)\ddot{\theta} = +mgl \sin \theta$$

(Not SHM - not the same sign)

### T.7 Isochrone



$$(b) \text{ K. E.: } T = \frac{1}{2} m \dot{s}^2,$$

$$\text{P. E.: } V = mgy;$$

Equation of motion in  $s$ :

$$\ddot{s} = -g \frac{\partial y}{\partial s}$$

(a) To obtain the equation of motion in  $x$ ,

$$T = \frac{1}{2} \left[ \dot{x}^2 + \left( \frac{dy}{dx} \right) \dot{x}^2 \right]$$

$$V = mgy$$

( $y(x)$ , function of  $x$ ),

$$L = T - V$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \left[ \dot{x}^2 + \left( \frac{dy}{dx} \right) \dot{x}^2 \right]$$

therefore the equation of motion is:

$$\frac{d}{dt} \left\{ m \dot{x} \left[ \dot{x}^2 + \left( \frac{dy}{dx} \right)^2 \dot{x}^2 \right] \right\} = -mg \frac{\partial y}{\partial x} + \frac{1}{2} m \dot{x}^2 \cdot 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2}$$

(the 1st term on the right comes from  $\frac{\partial V}{\partial x}$ , 2nd term from  $\frac{\partial T}{\partial x}$ )

This gives:

$$\ddot{x} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] + 2 \dot{x} \frac{dy}{dx} \frac{d^2 y}{dx^2} \dot{x} = -g \frac{dy}{dx} + \dot{x}^2 \frac{dy}{dx} \frac{d^2 y}{dx^2}$$

or

$$\ddot{x} \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right] = -g \frac{dy}{dx} - \dot{x}^2 \frac{dy}{dx} \frac{d^2 y}{dx^2}$$

The way to understand this is to think of the case of  $\frac{d^2 y}{dx^2} \rightarrow 0$ . The above equation then reduces to  $\ddot{x} =$

$\frac{-g \frac{dy}{dx}}{1 + \left( \frac{dy}{dx} \right)^2}$ . In other words, the same as  $\ddot{s} = -g \sin \alpha$  or  $\ddot{x} = -g \sin \alpha \cos \alpha$ , the equation of motion on a inclined plane.

Re-arranging,

$$\ddot{x} = \frac{-\frac{dy}{dx} \left( g + \dot{x}^2 \frac{d^2 y}{dx^2} \right)}{1 + \left( \frac{dy}{dx} \right)^2}$$

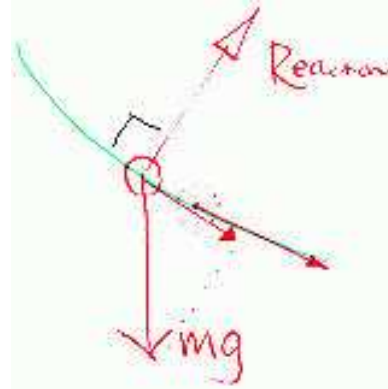
Remember that

$$\left| \frac{\frac{d^2 y}{dx^2}}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}} \right| = \frac{1}{r}$$

where  $r$  is the radius of curvature. The extra term is thus equal to  $v^2 (= \dot{x}^2 [1 + \left( \frac{dy}{dx} \right)^2])$  times  $1/r$

( $= \left| \frac{-\frac{d^2 y}{dx^2}}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}} \right|$ ) times  $\sin \alpha (= \frac{\frac{dy}{dx}}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2}})$ . i.e. the component parallel to  $x$  of a "centripetal force".

This force arises from the fact that the particle is constrained to follow the bends in the track and hence the difference between the normal component of gravity and the reaction from the plane must be such that it provides the necessary central force to accelerate the particle towards the centre of curvature (i.e. normal to the plane) at the correct rate. Note that even in a curved track, the force parallel to the plane is still just  $g \cos \theta$ , the same as in the inclined plane, in the absence of friction.



To get SHM, we need to see the equation of motion reduces to  $\ddot{s} \propto -s$ , and that means  $\frac{\partial y}{\partial s} \propto y$ . Therefore  $y$  must satisfy:  $y(s) = \frac{1}{2a} s^2$ , where  $a$  is a constant yet to be determined.

This gives

$$\left( \frac{dx}{dy} \right)^2 + 1 = \left( \frac{ds}{dy} \right)^2 = \left( \frac{a}{s} \right)^2 = \frac{a}{2y}$$

or

$$\frac{dx}{dy} = \pm \sqrt{\frac{a}{2y} - 1}$$

i.e.

$$x = \pm \int \sqrt{\frac{a}{2y} - 1} dy$$

The integration can be achieved by substituting  $\frac{a}{2y} = \frac{1}{\cos^2 \theta}$ :

$$\frac{x}{a} = \int |\tan \theta| \cos \theta \sin \theta d\theta$$

i.e.

$$\frac{x}{a} = \int \sin^2 \theta d\theta = \int \frac{1}{2} (1 - \cos(2\theta)) d\theta = \frac{1}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right]$$

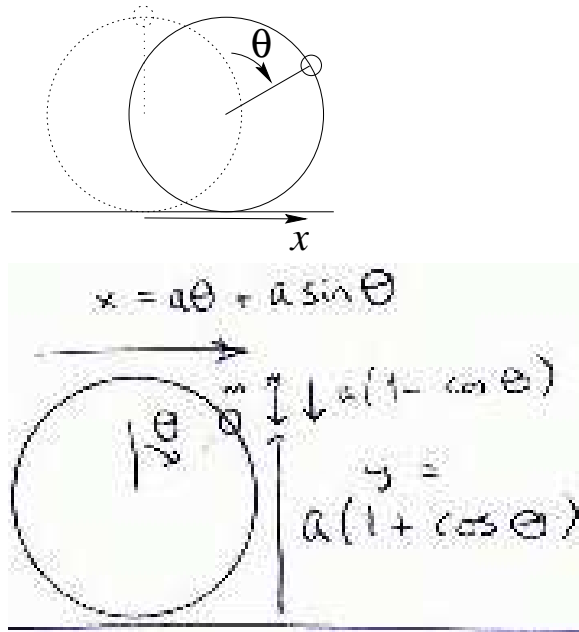
Finally, the answer is given by (using  $\alpha = 2\theta$ ):

$$\begin{cases} x + c = \frac{a}{4} \left[ \frac{\alpha}{2} \sin \alpha \right] \\ y = \frac{1}{4} [1 + \cos \alpha] \end{cases}$$

This is called a cycloid — the curve traced by a dot on the rim as a wheel rolls along.

(There are many equivalent answers, given that  $\alpha$  is just a dummy parameter and can well be replaced by e.g.  $(\frac{\pi}{2} - \alpha)$  or  $\frac{\alpha}{2}$ .)

### T.8 Hoop.



$$x = a\theta + a \sin \theta,$$

$$y = a(1 + \cos \theta)$$

(note the rolling advancement  $a\theta$  in  $x$ ). Therefore

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \dots = ma^2 \dot{\theta}^2 [1 + \cos \theta]$$

and

$$V = mgy = mga(1 + \cos \theta)$$

So we have

$$L = T + V = ma^2 \dot{\theta}^2 (1 + \cos \theta) - mga(1 + \cos \theta)$$

Written in  $s$ , the path length ( $ds^2 = dx^2 + dy^2$ ),  $T = \frac{1}{2} m \dot{s}^2$  is quite obvious.

For  $V$ , we have to rewrite  $y(s)$  in  $s$ . Luckily,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= a^2 ((1 + \cos \theta)^2 + \sin^2 \theta) \\ &= a^2 (2 + 2 \cos \theta) = 4a^2 \cos^2 \frac{\theta}{2} \end{aligned}$$

We have  $ds = 2a \cos \frac{\theta}{2}$ , or, integrating,  $s = 4a \sin \frac{\theta}{2}$ ;

$$\begin{aligned} y &= a(1 + \cos \theta) \\ &= 2a \cos^2 \frac{\theta}{2} \\ &= 2a \left( 1 - \frac{s^2}{(4a)^2} \right) \end{aligned}$$

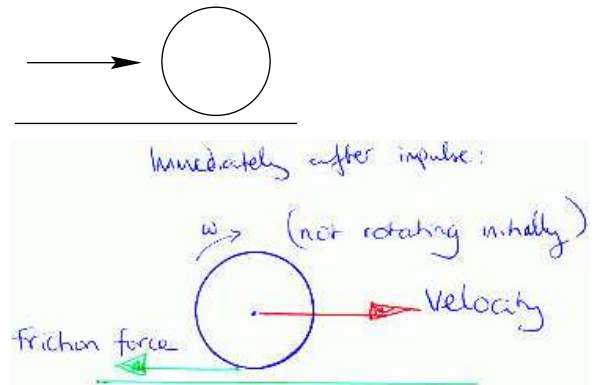
$L$  is given by

$$L = \frac{1}{2} m \dot{s}^2 + \frac{mgs^2}{8a} - 2mga$$

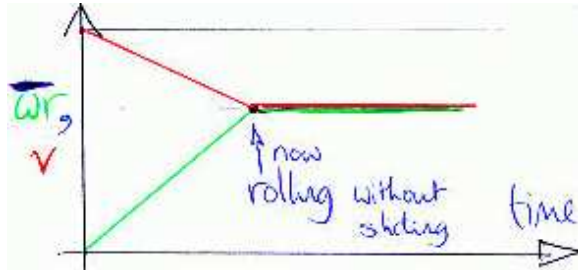
. Note the last constant has no consequence and that the motion is not SHM - the sign is different.

Also note the fact that this Lagrangian is exactly the same as that in the previous question, except for a sign change in  $y$  — In fact the curve traced out here is just upside down compared to the solution to the previous question.

### T.9 Snooker



If hit along the centre, the ball would shoot forward *gliding and slipping*. The friction decelerates the linear motion, while making it rotate. So it rotates faster and faster, while gliding along slower and slower. At some point later, when the linear velocity has slowed down enough and the rotation has speeded up enough such that they are compatible, the ball would roll instead of glide/slip.



$$m\dot{v} = -F \longrightarrow v = v_0 - \frac{F}{m}t$$

$$I\dot{\omega} = Fr \longrightarrow r\omega = 0 + \frac{Fr^2}{I}t = \frac{5F}{2m}t$$

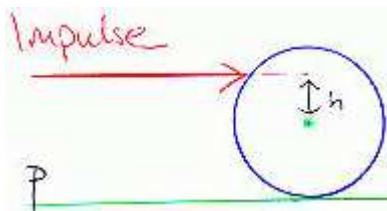
After  $t_0 = \frac{2mv_0}{7F}$ ,  $v = \omega r$ , at which point it starts rolling.

To make it roll right away, the momentum impulse  $\Delta p$  should be such that it sets up compatible linear motion and rotation together. Therefore

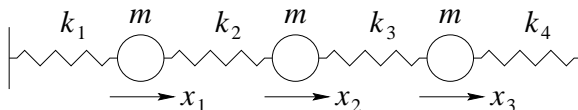
$$h\Delta p = I\omega = \frac{2}{5}ma^2\omega$$

$$\Delta p = mv$$

Using  $v = a\omega$  we obtain  $h = \frac{2a}{5}$  i.e. hit  $\frac{7a}{5}$  above table, or 70% the full height of the ball.



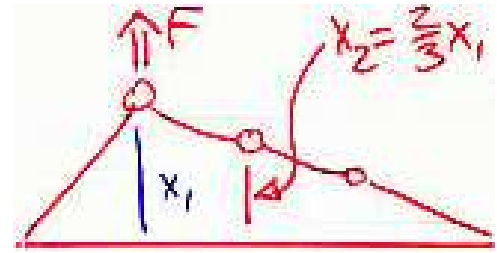
### T.10 Displaced springs



Potential  $V = \frac{1}{2}x_i K_{ij} x_j$ , where

$$K_{ij} = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{pmatrix}$$

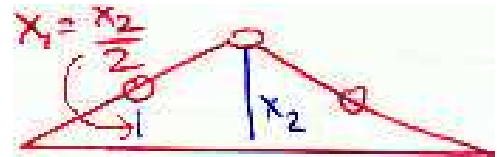
Assuming all the  $k_i$ 's are the same ( $= k$ ),



$$F_1 = (kx_1 + k(x_1 - x_2)) = \frac{4}{3}kx_1$$

so

$$x_2 = \frac{2}{3}x_1 = \frac{F_1}{2k}$$



$$F_2 = (k(x_2 - x_1) + k(x_2 - x_3)) = kx_2$$

and

$$x_1 = \frac{x_2}{2} = \frac{F_2}{2k}$$

So indeed  $x_j(F_i)$  has the same dependence as  $x_i(F_j)$ .

For the general case, the resisting force to external influence is  $F_i = -\frac{\partial V}{\partial x_i} = -K_{ij}x_j$ , so the external force is  $w_i = -F_i = K_{ij}x_j$ . If it is a stable dynamical system (i.e. it doesn't fly apart for the least force you apply onto any part of it), the  $K_{ij}$  should be invertible to give  $x_i = (K^{-1})_{ij}w_j$ . In fact, for coupled spring system,  $K_{ij}$  is symmetrical, and so is  $(K^{-1})_{ij}$  (property of symmetry matrix). Hence  $x_i = (K^{-1})_{ij}w_j$  is related to  $x_j = (K^{-1})_{ji}w_i$  by the same ratio  $(K^{-1})_{ij} = (K^{-1})_{ji}$ .

### T.11 Summation convention

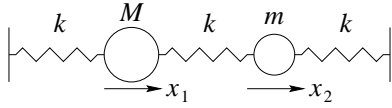
$$\begin{aligned} \text{grad } \underline{x}^T \underline{x} &= \frac{\partial}{\partial x_j} x_i x_i \\ &= \frac{\partial x_i}{\partial x_j} x_i + x_i \frac{\partial x_i}{\partial x_j} \\ &= 2\delta_{ij} x_i = \underline{2x_j} \end{aligned}$$

$$\begin{aligned} \text{grad } \underline{x}^T \underline{M} \underline{x} &= \frac{\partial}{\partial x_j} x_i M_{ik} x_k \\ &= \delta_{ji} M_{ik} x_k + x_i M_{ik} \delta_{jk} \\ &= (M_{jk} + M_{kj}) x_k \\ &= 2M_{jk} x_k, \quad \text{if } M_{jk} \text{ is symmetrical} \end{aligned}$$

$$\begin{aligned}\text{grad} \cdot \underline{\mathbf{x}} &= \frac{\partial}{\partial x_i} x_i \\ &= \delta_{ii} \\ &= \sum 1 = \underline{N},\end{aligned}$$

$N = 3$ , the dimension of the unit matrix  $\delta_{ij}$ .

### T.12 Two masses.



$$V = \frac{1}{2} \underline{\mathbf{x}}^T \underline{\mathbf{K}} \underline{\mathbf{x}}, \quad \underline{\mathbf{K}} = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$T = \frac{1}{2} \underline{\mathbf{x}}^T \underline{\mathbf{M}} \underline{\mathbf{x}}, \quad \underline{\mathbf{M}} = \begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix}$$

Equation of motion:  $\underline{\mathbf{M}} \ddot{\mathbf{x}} = -\underline{\mathbf{K}} \mathbf{x}$

Normal modes  $\underline{\mathbf{e}}$  satisfy  $\omega^2 \underline{\mathbf{M}} \underline{\mathbf{e}} = \underline{\mathbf{K}} \underline{\mathbf{e}}$ . i.e. need to solve eigen-equation  $|\underline{\mathbf{K}} - \lambda \underline{\mathbf{M}}| = 0$  for  $\lambda = \omega^2$ .

The equation ( $\lambda = \omega^2$ ) is:

$$\lambda^2 - 2k \left( \frac{1}{M} + \frac{1}{m} \right) + \frac{3k^2}{Mm}$$

$$\omega^2 = \frac{k}{M} + \frac{k}{m} + \sqrt{\left( \frac{k}{M} + \frac{k}{m} \right)^2 - \frac{3k^2}{Mm}}$$

It should be quite clear that the two normal modes are in-phase motion and out-of-phase motion respectively.

The special cases: if  $m \ll M$ , the two modes are :

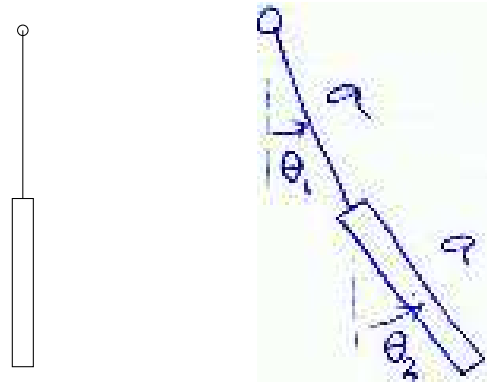
(1) the big one not moving much, while the small one oscillating between the two springs on either side of it,  $\omega^2 = \frac{2k}{m}$  (out of phase, actually the big one is moving slightly in the opposite direction).



(2) the big one moving around,  $\omega^2 = \frac{3k}{2M}$ , dragging the small one along in the middle of a long double-length spring (in phase motion). Note: if you get  $\omega^2 = 0$  for this, you should go back and keep the small terms of the next order!



### T.13 Baton.



The linear velocity of the c.g. is the vector sum of the two swinging motions, but since the angle is so small they are almost parallel and so just adds:

$$T = \frac{1}{2} m [a\dot{\theta}_1 + \frac{a}{2}\dot{\theta}_2]^2 + \frac{1}{2} I \dot{\theta}_2^2$$

where  $I = \frac{ma^2}{12}$ .

$$V = mga[(1 - \cos \theta_1) + \frac{1}{2}(1 - \cos \theta_2)]$$

Using the small angle approximation  $\cos \theta \approx 1 - \theta^2/2 + \dots$ , we get

$$V = \frac{1}{2} mga[\theta_1^2 + \frac{\theta_2^2}{2}]$$

Need to solve  $|\underline{\mathbf{K}} - \lambda \underline{\mathbf{M}}| = 0$ , where

$$\underline{\mathbf{K}} = mga \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\underline{\mathbf{M}} = ma^2 \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

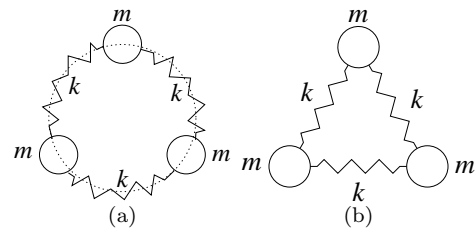
i.e.

$$\lambda^2 - 10\lambda + 6 = 0$$

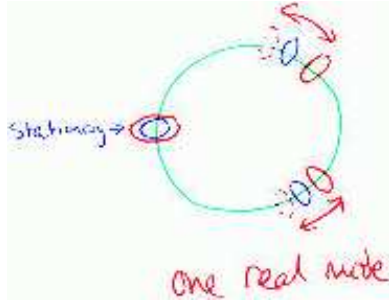
Solving this, we obtain:

$$\omega^2 = (5 \pm \sqrt{19}) \frac{g}{a}$$

### T.14 Triangles



(a) There are 3 degrees of freedom. One mode is the free rotation. And two other modes ( $\omega^2 = \frac{3k}{m}$ ) of the form:



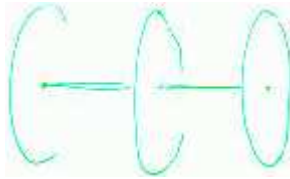
Note that even though there is a 3-fold symmetry, there are only two (degenerate) modes of this form, as the 3rd is just a linear combination (addition, actually) of the two.

(b) 6 degrees of freedom. There are 2 (degenerate) translational motions and a free rotation. There is one mode called the breathing mode in which the masses just go in and out in phase:



There are two degenerate (i.e. same frequency, and related by symmetry) modes corresponding to much the same two modes in (a), in which the triangle changes from being fat-short to narrow-tall. You would have thought there are three; but in fact the 3rd one is a linear combination of the other two.

(c)



On a linear shaft, there are three modes, one in which they all go in phase, another in which the two end ones go in opposite while the middle one is stationary; the 3rd mode is such that the middle one is going opposite to the two outer ones. The algebra is exactly the same as three masses connected with 2 springs. The frequencies are:

$$(1, 1, 1) : \omega^2 = 0$$

$$(1, 0, -1) : \omega^2 = \frac{k}{m}$$

$$(1, -2, 1) : \omega^2 = \frac{3k}{m}$$

**T.15 driven coupled masses** It is easier to understand by first looking at how we would have done it with a one-mass system.

$$m\ddot{x} = f \sin(\omega t) - 2kx$$

or,

$$f \sin(\omega t) = m\ddot{x} + 2kx$$

and the steady state solution substituting a trial solution  $x = x_0 \exp(i\omega t)$  and then solve for the amplitude. It is given by:  $x = \frac{f \exp(i\omega t)}{m(\omega_0^2 - \omega^2)}$  where  $\omega_0^2 = \frac{2k}{m}$ .

In the coupled mass system, you get:

$$m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} f \sin(\omega t) \\ 0 \end{pmatrix} - k \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

$$\begin{pmatrix} f \sin(\omega t) \\ 0 \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + k \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

And you just have to decouple the two equations in  $x_1, x_2$ . The answer will similarly, looks a bit like  $f$  driving two frequencies (the two normal modes) and depending on how close the driving frequency is to the normal mode frequencies, i.e. the expected answer should be  $x_1, x_2 = (f/m)[G(t)/(\omega_1^2 - \omega^2) + H(t)/(\omega_2^2 - \omega^2)]$  where  $G(t)$  and  $H(t)$  are some combination of exponential functions.

While it is possible (but tedious) to just use a trial substitution  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} \exp(i\omega t)$  and invert

the matrices by hand to obtain  $\begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}$  as a function of  $f$ , the procedure of decoupling is easier via the (known) eigen-solutions of the free oscillation:

$$\underline{\mathbf{e}}_1 = (1, 1), \omega_1^2 = \frac{k}{m}$$

$$\underline{\mathbf{e}}_2 = (1, -1), \omega_2^2 = \frac{3k}{m}.$$

First we'll re-write the above equation as:

$$\underline{\underline{\mathbf{M}}}^{-1} \begin{pmatrix} f \sin(\omega t) \\ 0 \end{pmatrix} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underline{\underline{\mathbf{M}}}^{-1} \underline{\underline{\mathbf{K}}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and we note that in the basis of  $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$ ,  $\underline{\underline{\mathbf{M}}}^{-1} \underline{\underline{\mathbf{K}}} \propto \underline{\underline{\mathbf{I}}}$ . In fact if we rewrite  $\underline{\mathbf{f}}$  and  $\underline{\mathbf{x}}$  in basis  $\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$ , due to orthogonality, the separate component must satisfy:

$$(\underline{\underline{\mathbf{M}}}^{-1} \underline{\mathbf{f}})'_i = -\omega^2 \underline{\mathbf{x}}'_i + \omega_i^2 \underline{\mathbf{x}}'_i$$

As expected, it has the anticipated functional form:

$$\underline{\mathbf{x}}'_i = \frac{-1}{\omega^2 - \omega_i^2} (\underline{\underline{\mathbf{M}}}^{-1} \underline{\mathbf{f}})'_i$$



Since  $(\underline{\mathbf{M}}^{-1}\underline{\mathbf{f}})'_i = \frac{f \sin \omega t}{2m} \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$ ,

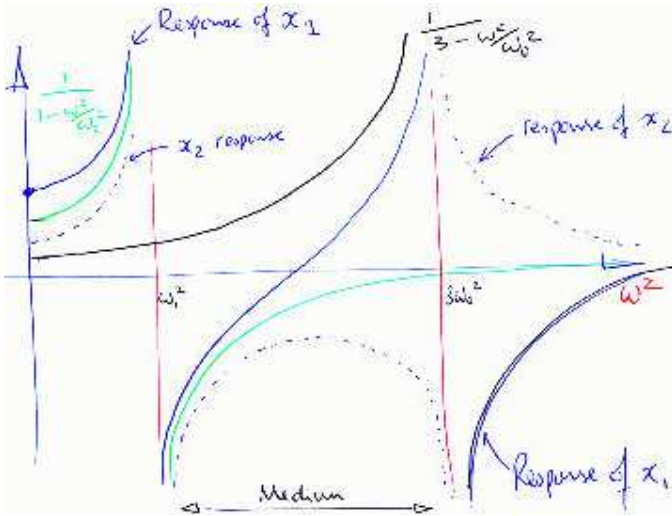
The answer is:

$$\underline{\mathbf{x}}'_i = \frac{-f \sin \omega t}{2m} \left[ \frac{1}{\omega^2 - \omega_1^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\omega^2 - \omega_2^2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]$$

as anticipated.

The response blows up when the driving frequency  $\omega$  is close to either of the normal mode frequencies,  $\omega_1 = \sqrt{\frac{k}{m}}$  or  $\omega_2^2 = \sqrt{\frac{3k}{m}}$ .

Interesting enough, at a particular frequency  $\omega$ , the response  $x_1$  (the mass under the driving force) is zero. This occurs when  $\omega^2 = (\omega_1^2 + \omega_2^2)/2$  in this case, but between the two resonance frequencies in the general case of unequal masses.



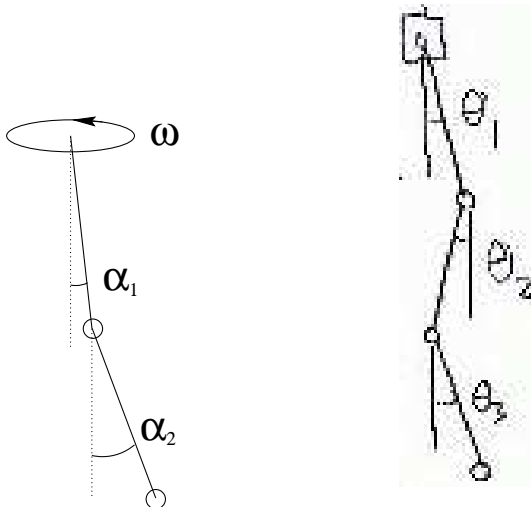
Note interesting phase relations:

Low frequency:  $x_1, x_2$  in phase with  $f$ .

Medium frequency:  $x_2$  anti-phase,  $x_1$  anti-phase, then flipped to be *in-phase* half-way.

High frequency:  $x_1$  anti-phase,  $x_2$  in phase with  $f$ .

### T.16 Double Pendulum



The small angle approximation  $\cos \theta \approx 1 - \theta^2/2 + \dots$  is needed right at the beginning; otherwise the algebra gets unmanagable ...

$$V = mga(1 - \cos \theta_1) + mga[(1 - \cos \theta_1) + (1 - \cos \theta_2^2)] \\ = 2 \times \frac{\theta_1^2}{2} + 1 \times \frac{\theta_2^2}{2}$$

$T =$  radial motion + circular/tangential motion

$$= \frac{m}{2} [\dot{\theta}_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2] + \frac{ma^2\omega^2}{2} [\theta_1^2 + (\theta_1 + \theta_2)^2]$$

For a general chains of  $n$ ,  $y_i = \sum_1^i \frac{\theta_i^2}{2}$ ,

$$V = mg \sum_i y_i = mg \sum_i (N - i + 1) \frac{\theta_i^2}{2}$$

and

$$T = m \sum_{i=1}^N \frac{1}{2} \left( \sum_{j=1}^i \dot{\theta}_j \right)^2 + a^2 \omega^2 \sum_{i=1}^N \frac{1}{2} \left( \sum_{j=1}^i \theta_j \right)^2$$

It follows that:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 2\omega^2 - 2 & +\omega^2 \\ \omega^2 & \omega^2 - 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

or inverting,

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \omega^2 - 2 & 1 \\ 2 & \omega^2 - 2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

In the special case of  $\omega = 0$ , eigenvalues ( $\lambda = -\Omega^2$ ) for  $\begin{pmatrix} -2 & 1 \\ 2 & -2 \end{pmatrix}$  are,

$$\underline{\mathbf{e}}_1 = (1, \sqrt{2}), \lambda_1 = -\Omega_1^2 = -2 + \sqrt{2}$$

$$\underline{\mathbf{e}}_2 = (-1, \sqrt{2}), \lambda_2 = -\Omega_2^2 = -2 - \sqrt{2}$$

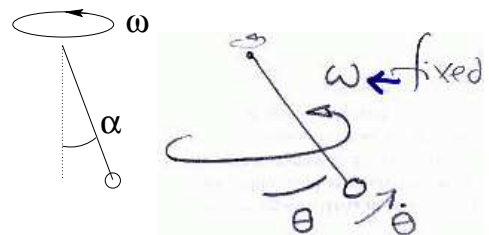
General case is similar, but with eigenvalues shifted by  $\omega^2$ :

$$\underline{\mathbf{e}}_1 = (1, \sqrt{2}), \lambda_1 = -\Omega_1^2 = -2 + \sqrt{2} + \omega^2$$

$$\underline{\mathbf{e}}_2 = (-1, \sqrt{2}), \lambda_2 = -\Omega_2^2 = -2 - \sqrt{2} + \omega^2$$

Or,  $\Omega_{1,2}^2 = 2 \mp \sqrt{2} - \omega^2$ . As  $\omega$  increases from zero,  $\Omega_1$  ( $\underline{\mathbf{e}}_1 = (1, \sqrt{2})$ ) will go critical first, at  $\omega^2 = 2 - \sqrt{2}$ .

(2nd part)



This problem can be tackled by two ways, (1) write down the Lagrangian exactly,

$$L = \frac{1}{2}ma^2(\dot{\theta}^2 + \sin^2 \theta \omega^2) + mga \cos \theta$$

and get an exact equation of motion, find its fixed points, and Taylor expand the equation of motion about those points, or (2) alternatively, approximate the Lagrangian at the outset by Taylor-expansion in  $\theta$  about  $\theta = 0$ .

General eqn. of motion:

$$T = \frac{ma^2}{2}[\dot{\theta}^2 \omega^2 \sin^2 \theta]$$

$$V = -mga \cos \theta$$

$$\frac{\partial L}{\partial \theta} = ma^2 \dot{\theta}$$

$$\frac{d}{dt}a^2 \dot{\theta} = \frac{\partial}{\partial \theta}[\sin^2 \theta \omega^2 \frac{a^2}{2} + ga \cos \theta]$$

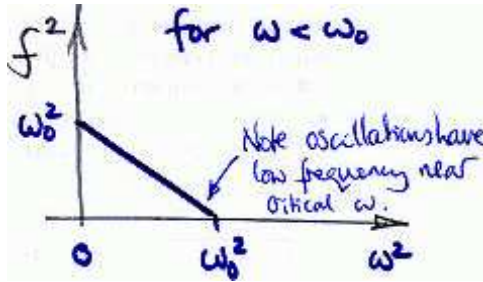
$$a^2 \ddot{\theta} = \sin \theta \cos \theta \omega^2 a^2 - ga \sin \theta$$

i.e.

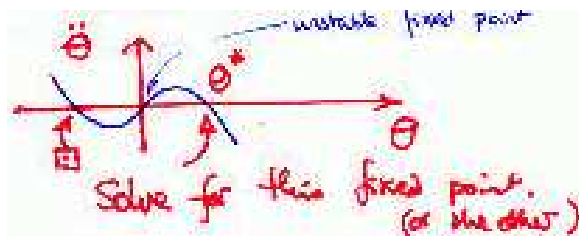
$$\ddot{\theta} = \sin \theta [\omega^2 \cos \theta - \omega_0^2]$$

with  $\omega_0^2 = \frac{g}{a}$ .

If  $\omega < \omega_0$  then SHM with  $\Omega^2 = (\omega_0^2 - \omega^2)$  about  $\theta = 0$ .



However, if  $\omega > \omega_0$ , then:

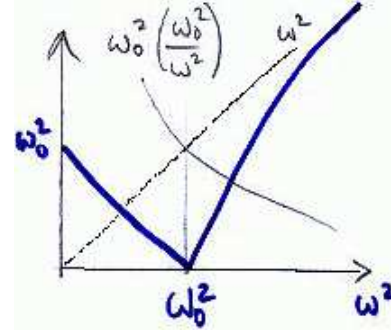


Solve for the non-zero fixed points:

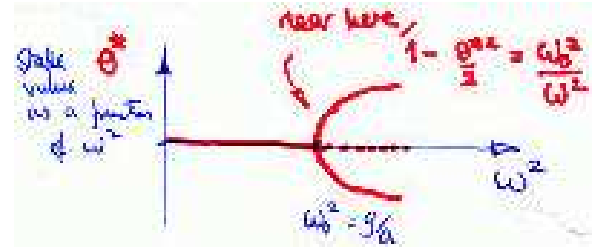
$$\cos \theta^* = \frac{\omega_0^2}{\omega^2}$$

Then we have SHM about  $\theta^*$ , with frequency  $\Omega^2$  given by:

$$\begin{aligned} \Omega^2 &= - \frac{\partial}{\partial \theta} \sin \theta [\omega^2 \cos \theta - \omega_0^2] \Big|_{\theta=\theta^*} \\ &= \omega^2 \sin^2 \theta^* \\ &= \omega^2 \left\{ 1 - \left( \frac{\omega_0^2}{\omega^2} \right)^2 \right\} \\ &= \omega^2 - \omega_0^2 \left( \frac{\omega_0^2}{\omega^2} \right) \end{aligned}$$



The behaviour is a bifurcation at  $\omega = \omega_0$ , as shown below in the state space diagram:



The alternative 2nd approach below illustrates some generality of bifurcation behaviour.

If we define  $\alpha(\omega) = \omega_0^2 - \omega^2$ , and  $\beta(\omega) = \frac{\omega^2}{2} - \frac{\alpha(\omega)}{6}$ , then we obtain from Taylor expanding  $\sin^2 \theta$  and  $\cos \theta$ :

$$L = \frac{1}{2}\dot{\theta}^2 - \frac{\alpha}{2}\theta^2 - \frac{\beta}{4}\theta^4$$

Note that  $\alpha(\omega)$  changes sign at  $\omega \sim \omega_0$ , while  $\beta$  is slow-varying around this region.

$$\ddot{\theta} = \frac{\partial L}{\partial \theta} = -\alpha\theta - \beta\theta^3 = \theta(-\alpha - \beta\theta^2)$$

This expression is zero for  $\theta = 0$  and  $\theta^2 = \frac{-\alpha}{\beta}$ . The former is stable when  $\alpha > 0$ , while the latter for  $\alpha < 0$ .

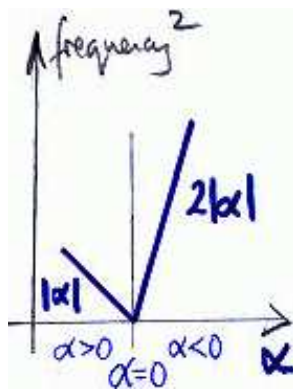
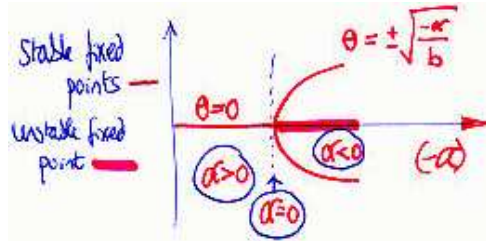
Frequency  $\Omega$  of small oscillations below critical point:

$$\Omega_-^2 = - \frac{\partial \ddot{\theta}}{\partial \theta} \Big|_{\theta=0} = +\alpha$$

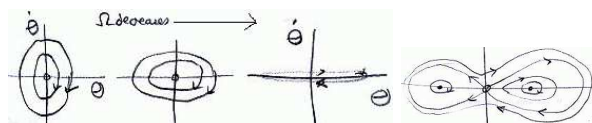
, while above the critical point,

$$\Omega_+^2 = - \left. \frac{\partial \ddot{\theta}}{\partial \theta} \right|_{\theta=\theta^*=\sqrt{-\frac{\alpha}{b}}} = 2\beta\theta^2 = -2\alpha$$

So generally,  $\Omega^2$  is  $|\alpha|$  below the critical point, and  $2|\alpha|$  above it, at a bifurcation.



The state space diagram progresses as follows from low  $\omega$  to high  $\omega$ :



#### T.17 Shear Strain - notes for an essay:

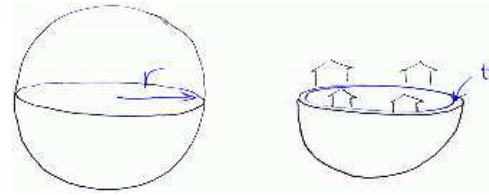
Explanation should give examples, making an explicit change of basis and showing (as in lectures) what a shear strain looks like in another basis.

A definition of shear strain would be a good idea. A sketch showing how distances changes with a shear is instructive.

Distances are unchanged in the directions of N, S, E, W; they increases in the pink (NE-SW) sectors and decreases in the green (NW-SE) sector.

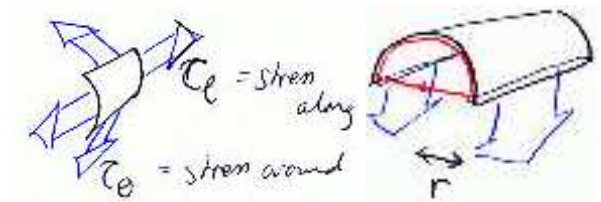
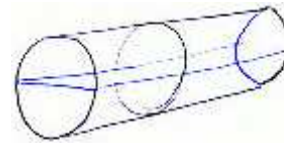


#### T.18 Sphere.



Total force exerted by stress  $= 2\pi r \cdot t \cdot \tau = P \cdot \pi r^2$ , gives  $P = \tau \cdot \frac{2t}{r}$ , or inverting,  $\tau = P \frac{r}{2t}$ .

(Check dimensions: stress  $\tau$  has the same dimension as pressure  $P$ ).



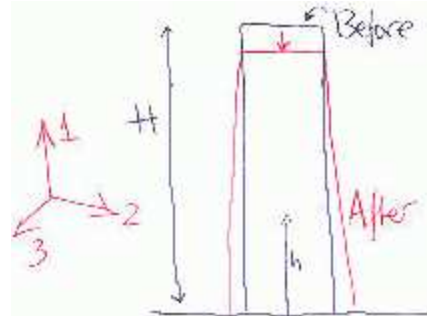
$\tau_\theta \cdot 2t = P \cdot 2r$ , giving  $\tau_\theta = P \frac{2r}{2t} = P \frac{r}{t}$ , twice as big.



$2\pi r t \tau_\rho = \pi r^2 P$ ,  $\tau_\rho = P \frac{r}{2t}$ . Same as sphere.

Practical implication: For a tubular container with end caps, as the pressure inside increases, it will go critical and burst due to  $\tau_\theta > \tau_c$ , and a hemispherical end cap is simply over-kill as it is twice as strong as necessary. To save material costs, etc, it is usual to have  $r_{\text{cap}} = 2r_{\text{tube}}$  so that the end cap is equally strong. c.f. the shape of the container at the back of common lorries used for transporting petrol; or shape of compressed air cylinders.

**T.19 Pillar.** Stress and strain increases towards the bottom. Pillar shortens because its base is compressed by its weight. It also thickens because of its Poisson's ratio.



The stress tensor is

$$\underline{\underline{\tau}}(h) = \begin{bmatrix} \tau_{11}(h) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where  $\tau_{11}(h) = -(H - h)\rho g$ .

The resulting strain is

$$\underline{\underline{\mu}}(h) = \begin{bmatrix} \frac{\tau_{11}(h)}{Y} & 0 & 0 \\ 0 & \frac{-\sigma\tau_{11}(h)}{Y} & 0 \\ 0 & 0 & \frac{-\sigma\tau_{11}(h)}{Y} \end{bmatrix}$$

Total change in height:

$$\begin{aligned} -h_1 &= \int_0^H dh \mu(h) \\ &= - \int_0^H dh (H - h) \frac{\rho g}{Y} \\ &= - \frac{H^2}{2} \frac{\rho g}{Y} \end{aligned}$$

After flooding, there is an additional stress exerted in the 22 and 33 directions.

Now, pressure  $P(h) = \rho_{\text{water}}g(H - h)$ . Here we neglect the different between the natural height and the compressed upright height.

The stress adds linearly, and the strain also adds linearly, so we can find the response by superposition. The extra terms due to flooding are:

$$\Delta \underline{\underline{\tau}}(h) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -P(h) & 0 \\ 0 & 0 & -P(h) \end{bmatrix}$$

$$\underline{\underline{\mu}}(h) = \begin{bmatrix} +2\frac{\sigma P(h)}{Y} & 0 & 0 \\ 0 & (\dots) & 0 \\ 0 & 0 & (\dots) \end{bmatrix}$$

where  $(\dots)$  is irrelevant to our present task.

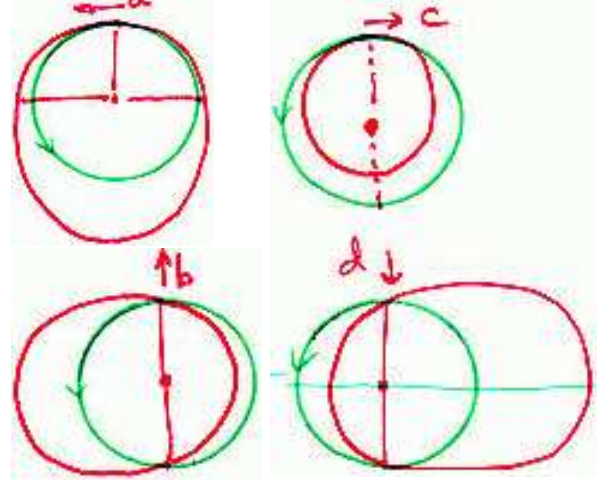
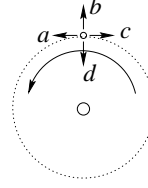
So the integrated strain resulting from the water is:

$$\begin{aligned} \Delta h &= \int dh \Delta \underline{\underline{\mu}}(h) \\ &= \int dh 2\sigma\rho_w g(H - h) \\ &= \frac{H^2}{2} \frac{2\sigma\rho_w g}{Y} \end{aligned}$$

Therefore,  $-h_2 = \frac{H^2 g}{2Y} [2\rho_w \sigma - \rho]$

[Ans:  $h_1 = \rho g H^2 / 2Y$ ,  $h_2 = (\rho - 2\sigma\rho_w)gH^2 / 2Y$ ]

## T.20 Ellipses.



Period  $\propto a^{3/2}$ . It is possible to prove for  $V = -\frac{A}{r}$  and  $F = -\frac{A}{r^2}$  that the semi-major axis  $a = -\frac{A}{2E} \propto |E|^{-1}$  (remember  $E$  is negative, so when  $E$  increases,  $|E|$  decreases in magnitude), and the semi-latus-rectum  $r_0 = \frac{J^2}{Am} \propto J^2$ .

When the impulse is radial, it doesn't affect the angular momentum, so the semi-latus-rectum is unchanged; therefore the new orbit will pass through the same diametrical points.

If the 4 impulses are equal, then the following comparison can be made.

- Ellipses  $b$  and  $d$  have equal semi-major axis, quadratically larger than the original radius (because everything changes as the square of the impulse).
- Ellipses  $a$  and  $c$  have semi-major radius that is linear in the impulse ( $dE \sim \frac{1}{2}d(v^2) \sim vdv$ ).

Many of the irrelevant results are:

$$a = \frac{r_0}{1 - e^2}, b = \frac{r_0}{\sqrt{1 - e^2}}, r_{\min} = \frac{r_0}{1 + e}$$

**T.21 Daylength.** The question was phrased ambiguously: Day length could refer to the exact duration of a roughly 24-hour period from mid-day to mid-day. And it could also refer to the period from sunrise to sunset (i.e.  $\sim 12$  hours). For the 1st part "how does the elliptical orbit affect daylength" the first meaning is intended. Dr MacKay's web page (<http://wol.ra.phy.cam.ac.uk/teaching/dynamics/>)

has the link to the full details about the equation of time.

If daylength means the time between sunrise and sunset, the change in distance has a very negligible effect on daylength, as follows:



$$\theta \approx \frac{2r_e}{R} = \frac{10^7}{10^{11}} = 10^{-4}$$

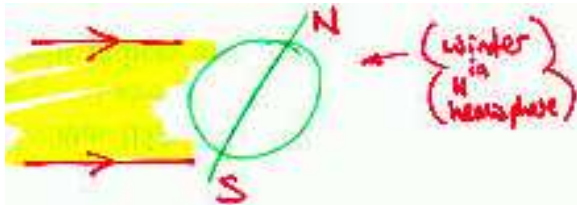
$$\frac{\delta\theta}{\theta} = -\frac{\delta R}{R} \sim -\frac{1}{30}$$

So  $|\delta\theta| = \frac{1}{30} \times 10^{-4} \text{ rad} \approx 3 \times 10^{-6} \text{ rad}$ , so the daylength at the equator changes by  $\frac{3 \times 10^{-6}}{2\pi} \times (1 \text{ day}) \approx \frac{1}{20} \text{ sec}$ .

However, it is true that at the winter solstice, sunrise continues to get later in the day ( $\approx$  half minute per day).

The time of sunrise and sunset depends on:

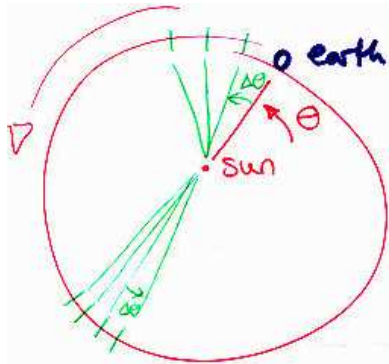
(a) the inclination of the sun to the N-S axis:



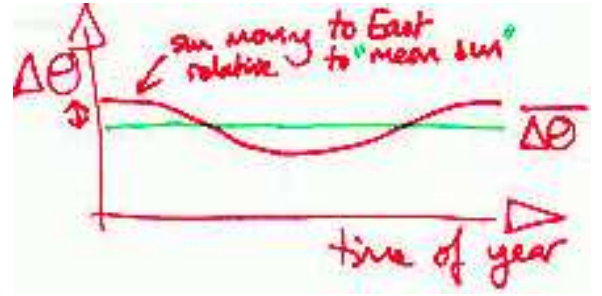
(b) the rotation of the earth relative to the fixed stars;

(c) the apparent rotation of the sun relative to the fixed stars.

At the winter solstice, effect (a) is at an extremum. Effect (b) is constant (one rotation per day). So any change in sunrise is produced by the non-uniformity of the apparent motion of the sun.



$r^2\dot{\theta} = \text{constant}$ , so  $\Delta\theta$  (per day) is greater at the perihelion (the closest point) than the aphelion.



$\Delta\theta = \dot{\theta} \cdot t = \frac{ct}{r^2}$ , and at the perihelion,  $\Delta\theta = \overline{\Delta\theta} + \delta\theta$ ,

$$\left| \frac{\delta\theta}{\overline{\Delta\theta}} \right| = 2 \left| \frac{\delta r}{r} \right|$$

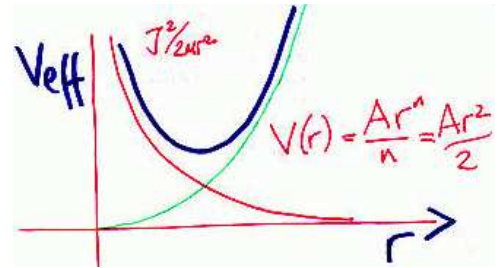
We are interested in  $\delta\theta$  translated into the time advance or delay that produces in sunrise.

$$\delta\theta = \overline{\Delta\theta} \cdot \frac{r_{\text{max}} - r_{\text{min}}}{\bar{r}} = \frac{2\pi}{365} \cdot \frac{1}{30}$$

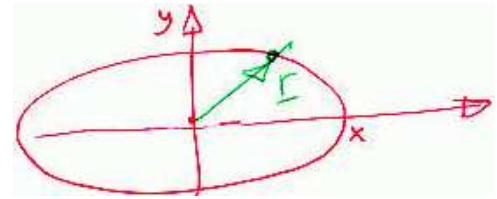
Time advance per day is this over  $2\pi$  of 24 hours,  $\sim 10$  seconds.

(This differs by factor of 3 with actual data - see links from webpage for full explanation.)

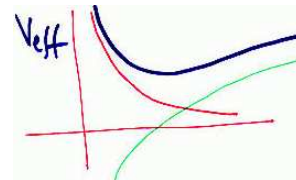
**T.22 Power law potentials.** The effective potential is expressed as:  $\frac{J^2}{2mr^2} + V(r)$ . (a)  $n = 1$ :  $V(r) = \frac{Ar^2}{2}$ ,



All motions are bounded; motion is elliptical, centred on the origin.

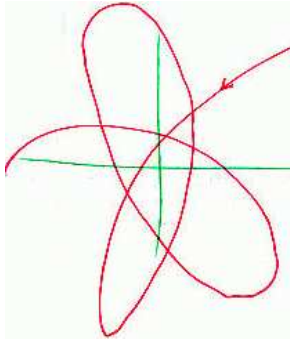


(b)  $n = -1$ :  $F = -\frac{A}{r}$ ,  $V(r) = -A \log r$

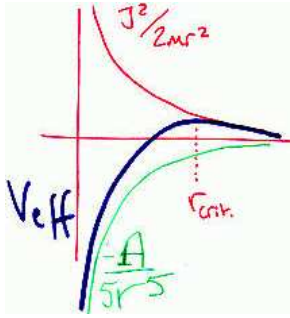


All motions are bounded, but "only just". Actual motions are not easy to describe. Maybe something like this:

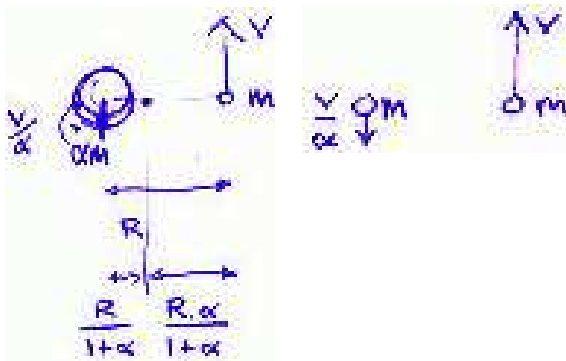




(c)  $n = -6$ :  $V(r) = -\frac{A}{5r^5}$ . Beyond a certain distance  $r_c$  and with sufficient outward velocity, it will depart and never come back. On the other hand, if  $r < r_c$  it will spiral and fall all the way into the centre.



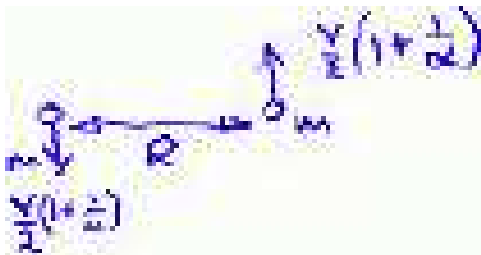
**T.23 Double star.** Note that after the explosion, the total momentum is non-zero. Before and after the explosion:



Condition for circular orbit (before):

$$\frac{Gam^2}{R^2} = mv^2 \frac{1+\alpha}{R\alpha} + (\alpha m) \left(\frac{v}{\alpha}\right)^2 \frac{1+\alpha}{R}$$

After the explosion, in the new zero momentum frame (going up  $\uparrow$  at  $\frac{v}{2} \left(1 - \frac{1}{\alpha}\right)$ ):



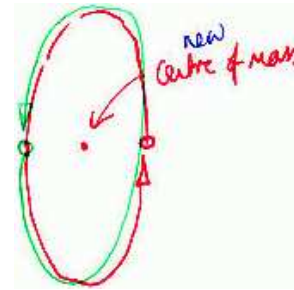
Total energy in this frame:

$$E = -\frac{Gm^2}{R} + m \left(\frac{v}{2}\right)^2 \left(1 + \frac{1}{\alpha}\right)^2$$

Using our previous circular orbit relation:

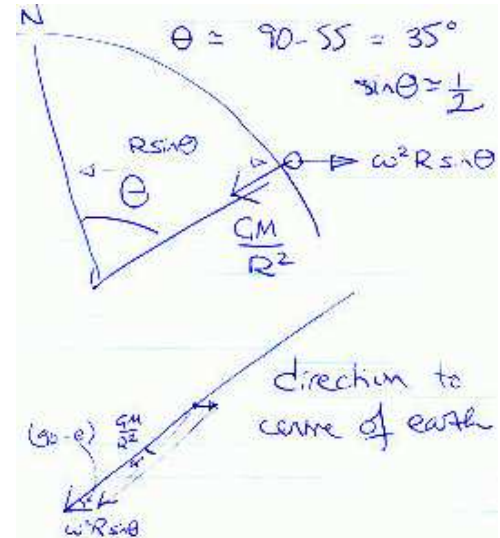
$$\begin{aligned} E &= \frac{Gm^2}{R} \left[ -1 + \frac{1}{4} \left(1 + \frac{1}{\alpha}\right) \frac{\alpha^2}{1+\alpha} \right] \\ &= \frac{Gm^2}{R} \left[ -1 + \frac{1}{4} (1 + \alpha) \right] \\ &= \frac{Gm^2}{4R} [\alpha - 3] \end{aligned}$$

Therefore if  $\alpha > 3$ , the total energy in the zero momentum frame is positive, and the two particles would have enough energy to escape from each other's influence. In the new zero momentum frame, it could have been some new elliptical orbit:



But only if the total energy, in the zero momentum frame, is negative; otherwise they fly apart.

**T.24 Vertical.** (a) Angle of deflection given by [ assuming  $\omega^2 R \ll \frac{GM}{R^2}$  ]:



$$\frac{\sin \alpha}{\omega^2 R \sin \theta} \approx \frac{\sin(90^\circ - \theta)}{\frac{GM}{R^2}}$$

$$\alpha \approx \sin \theta \cos \theta \frac{\omega^2 R}{\frac{GM}{R^2}} = \sin \theta \cos \theta \frac{\omega^2 R}{g_{\text{gravity}}}$$

$$\alpha = \frac{1}{2} \frac{\sqrt{3}}{2} \left( \frac{2\pi}{24 \times 3600} \right)^2 \frac{6 \times 10^6}{10} \\ \approx 3 \times 10^{-3} \text{ radians} \approx 0.2^\circ$$

(b) (i) Conservation of angular momentum:

$$r^2 \dot{\theta} = r_0^2 \Omega \implies \dot{\theta} = \frac{r_0^2}{r^2} \Omega$$

Now  $r = r_0 - \frac{1}{2}gt^2$ , so  $\frac{r}{r_0} = 1 - \frac{gt^2}{2r_0}$ , or taking approximation ( $\frac{gt^2}{2r_0} \ll 1$ ):  $\frac{r_0^2}{r^2} = 1 + \frac{gt^2}{r_0}$

Thus

$$\dot{\theta} = \left( 1 + \frac{gt^2}{r_0} \right) \Omega$$

therefore

$$\theta = \Omega \left( t + \frac{gt^3}{3r_0} \right)$$

The first term ( $\Omega t$ ) is simply the angular advancement of the ground below the helicopter, so the distance moved (to the east) is

$$x = r_0 \times \Omega \frac{gt^3}{3r_0} = \frac{\Omega gt^3}{3}$$

(b) (ii) Using Coriolis force argument:

$$\dot{z} = gt$$

$$\ddot{x} = 2\Omega \dot{z} = 2\Omega gt$$

integrating twice:

$$x = \frac{\Omega gt^3}{3}$$

[Ans: 24 cm to the East]

**T.25 Plane.** Coriolis force =  $2m\omega v$  to the right.

Distance displaced:  $t = \frac{D}{v}$ ,  $x = \frac{1}{2}at^2 = \omega \frac{D^2}{v} \approx 3\text{km}$  south of Cambridge for  $v = 100 \text{ mph} = 50 \text{ m s}^{-1}$ .

Note that the slower the plane the larger the deflection.

Additionally, alternative treatment (details in Dr Mackay's solution): the Coriolis force could be balanced by air drag eventually, and the plane drifts sideways at constant velocity. This constant sideways drift (after some calculations) is given by  $\frac{4\omega v D}{g} \approx \frac{1}{4}\text{km}$ .

**T.26 Spaceman.** Assuming the orbiting time is sufficiently large, the spaceman only need to cover about 100 yards (slightly corrected for orbiting motion); let's say the boot is one 50th of his weight (1.5 Kg vs. 75Kg), and he can throw it away at  $1 \text{ m s}^{-1}$ . Then he would recoil back at  $0.02 \text{ m s}^{-1}$ . So it would take him  $\sim 5000 \text{ s}$ , or one and a half hour to get back, if he does it accurately enough.

Viewed in an inertial frame, his path would be an arc of a (elliptic) orbit intercepting the circular one of the spaceship; but viewed in the rest frame of the ship, he would have to toss the boot in a direction not directly opposite to the ship to counterbalance any fictitious forces (centrifugal, Coriolis) he would experience.

*Sanity check: using the known radius of Earth and acceleration due to gravity, a spaceship orbiting on the surface of earth [assuming no mountains, air, etc] has a period of 5000s; the moon has a period of 27 days and a typical satellite probably has an orbiting period of a couple of days. So the correction to straight-line motion is although small, still significant and would be noticeable - we are talking about throw the boot maybe  $5^\circ$  to  $10^\circ$  off the line of direct vision.*

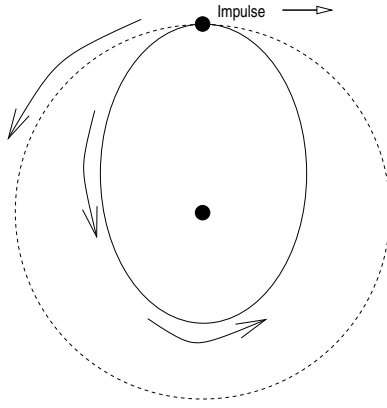
*Dr Mackay: There are several ways to throw the boot to get home.*

*I would like the students to understand the option of throwing the boot straight at the space station, which works if the man is ahead or behind the station in the orbit.*

The time scale with this strategy is rather inconvenient though - the spaceman would need to orbit in the opposite sense for at least hours (a substantial part of an orbiting period) before getting back onto the ship...

**T.27 Satellite.** A satellite in a circular orbit in potential  $V(r) = -\frac{GM}{r}$  per unit mass has kinetic energy given by  $T(r) = \frac{1}{2}mv^2$  and where  $\frac{mv^2}{r} = \frac{GM}{r^2} \implies T(r) = \frac{GM}{2r} = -\frac{1}{2}V(r)$ . If we move to a lower potential, the kinetic energy gets bigger. The total energy decreases, but half of that lost energy is turned into the kinetic energy.

First consider a brief impulse slowing the satellite.



The new orbit has two properties:

- After half a orbit, the satellite is going faster than before the impulse.
- After a quarter orbit, the satellite has an inward radial velocity.

Continuous version:

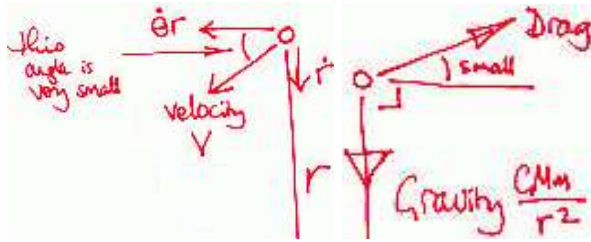
We anticipate the correct inward drift-rate, using:

$$E = \frac{1}{2}V(r) = -\frac{GMm}{2r}$$

$$\frac{dE}{dt} = -fv = \frac{GMm}{2r^2} \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = -\frac{2fv}{\left(\frac{GMm}{r^2}\right)} = -\frac{2f}{m\dot{\theta}}$$

Now draw the situation using  $(r, \theta)$  coordinates for both the velocity and the force:



The change in angular momentum is given by:

$$\frac{d}{dt}[m\dot{\theta}r^2] = -fr$$

$$\Rightarrow m\ddot{\theta}r^2 + 2m\dot{\theta}r\frac{dr}{dt} = -fr$$

$$\Rightarrow \ddot{\theta} = \frac{1}{r^2} \left( -\frac{fr}{m} - 2\dot{\theta}r\frac{dr}{dt} \right)$$

(The other equation is  $m\ddot{r} = 0$ , i.e. constant inward drift rate, by construction. . . )

$$\Rightarrow \ddot{\theta} = -\frac{f}{mr} - 2\frac{\dot{\theta}}{r}\frac{dr}{dt}$$

The first term is loss of angular velocity due to drag, while the second is gain of angular velocity because  $\frac{dr}{dt} < 0$ , or Coriolis force term arising from the inward drift. Note angular momentum does not conserve due to the drag force.

Substituting  $\frac{dr}{dt} = -\frac{2f}{m\dot{\theta}}$  we obtained earlier,

$$\Rightarrow \ddot{\theta} = -\frac{f}{mr} + \frac{4f}{mr} = \frac{3f}{mr}$$

The rate of increase of speed:

$$\begin{aligned} \frac{d}{dt}|v| &= \frac{d}{dt}|\dot{\theta}r| \\ &= \ddot{r} + \dot{\theta}\dot{r} \\ &= \frac{3f}{m} - \dot{\theta} \cdot \frac{2f}{m\dot{\theta}} \\ &= \frac{f}{m} \end{aligned}$$

i.e. speed increased.

**T.28 Space elevator.** Besides the compressional force due to the whole weight, there is possibly also a torque at the bottom to keep the structure in orbit. The top has a higher linear velocity than the bottom, so a torque is required to keep turning the structure. Therefore the tower is effectively being bended continuously sideways.

In fact, although the compressional stress is enormous, the structure, supposing that you can actually erect one up for one split second, will probably fail by bending under say, wind, rather than actually fracturing under the compressional stress.

“Centrifugal” force exactly balances gravity at the elevation of a geostationary orbit — i.e. the height of the tower would need to be a few times the radius of earth for “Centrifugal” force to help supporting its weight.

According to some databook, for steel:

Young’s modulus:  $E = 20.1 - 21.6 \times 10^{10}$  Pa

Shear modulus:  $G = 7.8 - 8.5 \times 10^{10}$  Pa

Poisson ratio:  $\sigma = 0.28 - 0.30$

Bulk modulus:  $k = 16.5 - 17.0 \times 10^{10}$  Pa

Compressibility:  $\chi = 0.61 - 0.59 \times 10^{-11}$  Pa<sup>-1</sup>

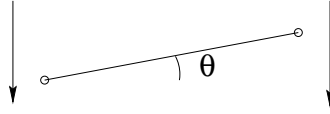
and  $\chi = \frac{1}{k}$ ,

$$E = 2G(1 + \sigma) = 3k(1 - 2\sigma)$$

For a 300km column, strain =  $\frac{\rho gh}{E} \approx 12\%$  at the base. This kind of strain is quite beyond the endurance of the material. Breaking strain of most materials is at most of the order of 1%, and usually much less. The exception being rubber or certain rubbery materials.

The tensile stress strength of steel is listed as  $5 \times 10^8$  Pa, i.e. the breaking strain is about 0.3%. c.f. tallest building maybe 150 floors of 2 m each  $\sim 300$  m tall.

**T.29 Bouncing rod or coin.** It is easiest to express everything in terms of the impulse  $P$  at the point of impact; this impulse changes the linear velocity and also set the rod rotating.



New velocity  $v$  (downward +ve):  $P = mu - mv$   
 New angular velocity  $\omega$ :  $I\omega = Pa$

Now, energy conservation in elastic collision:

$$\begin{aligned} \frac{1}{2}mu^2 &= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}m \left[ u - \frac{P}{m} \right]^2 + \frac{1}{2} \frac{I}{a^2} P^2 \end{aligned}$$

This gives (besides  $P = 0$ , i.e. no impact) a solution of  $P$  as:

$$P = \frac{2mu}{1 + \frac{ma^2}{I}}$$

From this we obtain:

$$v = \frac{\frac{ma^2}{I} - 1}{\frac{ma^2}{I} + 1} u = \frac{1 - \frac{I}{ma^2}}{1 + \frac{I}{ma^2}} u$$

and

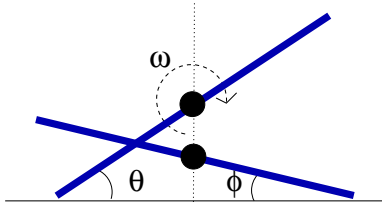
$$a\omega = \frac{Pa^2}{I} = \frac{2u}{\frac{I}{ma^2} + 1}$$

The instance of next contact is such that the amount of rotation will flip the opposite edge to hit the table, taking into account of the fact that the c.g. will have moved (down) by then.

From geometry:

Amount moved by c.g. =  $vt = a(\theta - \phi)$

rotation =  $\omega t = \theta + \phi$ ,



Therefore

$$\begin{aligned} \frac{\phi}{\theta} &= \frac{a\omega - v}{a\omega + v} \\ &= \frac{a\omega - v}{a\omega + v} \\ &= \frac{1 + \frac{I}{ma^2}}{3 - \frac{I}{ma^2}} \end{aligned}$$

For coin flipping sideways,  $I = \frac{1}{4}ma^2$ ,  $\frac{\phi}{\theta} = \frac{5}{11}$ , while for the special case of  $I = ma^2$ ,  $\frac{\phi}{\theta} = 1$ . The latter case corresponds to a situation in which two masses are attached by a massless middle cord, and the velocity of each mass simply reverses when it hit the table, with their c.g. instantaneous at rest and hanging mid-air for eternity after the initial contact, while the two end-masses bounces up and down alternately.

[Ans: for coin, next contact is at an angle of  $5\theta/11$ ; for the special case  $I = ma^2$ , the contact is at  $\theta$ .]

**T.30 Disc.** Moment of inertia of disc about  $\perp$  axis:

$$I_z = m \int_0^R \frac{2\pi r dr}{\pi R^2} r^2 = \frac{1}{2}mR^2$$

By  $\perp$  axis rule,  $I_x = I_y = \frac{1}{4}mR^2$ .

$$\underline{\omega} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{\mathbf{I}} = \begin{pmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{pmatrix}$$

$$\underline{L} = \underline{\mathbf{I}} \underline{\omega} = \frac{mR^2}{4\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0 \\ 1.4 \end{pmatrix} \times 10^{-3} \text{ kg m}^2 \text{ s}^{-1}$$

$$\text{K. E.} = \frac{1}{2} \underline{\omega} \cdot \underline{\mathbf{I}} \underline{\omega} = \frac{1}{2} \underline{\omega} \cdot \underline{L} = \frac{3}{16}mR^2 = \frac{3}{4} \text{ mJ}$$

Ans:  $(0.7, 0, 1.4) \times 10^{-3} \text{ kg m}^2 \text{ s}^{-1}$  w.r.t. obvious axes;  $\frac{3}{4} \text{ mJ}$

**T.31 Precession of earth.** In reality, the moment of inertia of the earth is such that  $\frac{I_3}{I_1} = 1 + \frac{1}{300}$ .

To estimate for this, we can assume that the earth is semi-solid (or flexible enough) so that after spinning for many many years, it would settle into an equilibrium shape governed by gravity and the "centrifugal force". The centrifugal force " $\omega^2 r$ " per unit mass would give a "pseudo-potential" of " $-\frac{\omega^2 r^2}{2}$ ". The surface of the earth would be

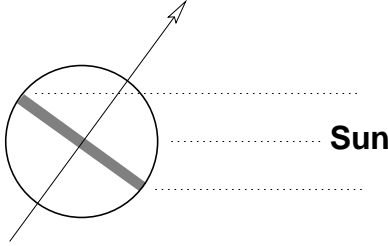
a “equipotential surface”, so the polar radius  $r_p$  would be related to the equatorial radius  $r_e$  by:

$$-\frac{GM}{r_e} - \frac{\omega^2 r_e^2}{2} = -\frac{GM}{r_p}$$

or, after a bit of simplification:  $\frac{\Delta r}{r} \approx \frac{r_e - r_p}{r_e} \approx \omega^2 r_e \cdot \frac{r_e r_p}{2GM} \approx \frac{\omega^2 r}{2g} = \frac{(2\pi/86400)^2 \times 6 \times 10^6}{2 \times 10} \approx \frac{1}{600}$

(From this,  $\frac{I_3}{I_1} = \frac{\sum(x^2 + y^2)}{\sum(y^2 + z^2)} \approx \frac{2r_e^2}{r_e^2 + r_p^2} = 1 + \frac{1}{2} \frac{\delta r^2}{r^2} = 1 + \frac{1}{600}$ . Oh well...)

Assuming the extra mass is all in a equatorial ring. There is a torque from the sun because the part of the ring closer to the sun is attracted by a bigger force than the part beyond, and the net result is a force in the ring centre plus a torque trying to turn the ring completely side on.



$$\frac{\frac{2}{5}mr^2 + \alpha mr^2}{\frac{2}{5}mr^2 + \frac{1}{2}\alpha mr^2} = \frac{301}{300} \Rightarrow \alpha = \frac{4}{5 \times 299} \approx 0.00268$$

Take a further approximation: the extra mass is concentrated in 4 lumps around the ring. The torque  $\tau$  is given by:

$$\begin{aligned} \tau &= \frac{\alpha m}{4} GM r \sin \theta \left( \frac{1}{d^2} - \frac{1}{(d + r \cos \theta)^2} \right) \\ &= \frac{\alpha m r^2 \sin \theta \cos \theta}{2} \Omega^2 \end{aligned}$$

where  $\frac{GM}{d^2} = \Omega^2 d$ , the revolution rate of the earth around the sun.

Note the torque  $\tau$  is (1)  $\propto \Delta I \sim \alpha m r^2$ , i.e. the equatorial bulge, (2)  $\propto \sin \theta \cos \theta$ , i.e. zero when the ring is  $\perp$  or  $\parallel$  to the direction of the sun, (3)  $\propto \Omega^2$ , remember the rate of change of the gravitation field is  $\frac{\partial}{\partial d} \left( \frac{GM}{d^2} \right) = \frac{2GM}{d^3} = 2\Omega^2$ . In fact  $\tau \propto \nabla \underline{\mathbf{g}}$  = gradient of gravitation field.

Rate of precession =  $\frac{\tau}{L \sin \theta}$ ,  $\theta = 23.5^\circ$ , and  $L = I\omega$  is the angular momentum for the spinning of the earth,  $I = \frac{2}{5}mr^2$ ,  $\omega = 2\pi$  per day. This gives  $1.7 \times 10^{-12} \text{ s}^{-1} = 0.5 \times 10^{-4}$  per year.

The actual precession rate is  $1.6 \times 10^{-4} \text{ year}^{-1}$ . The approximation to the ring geometry probably accounts for the discrepancy.

A more exact treatment would be to find the contribution to the  $\tau$  due to each part of the earth. It is given by the difference in the strength of the force to that in the centre ( $\underline{\mathbf{r}} \cdot \nabla g$ , in the direction of  $d$ ) multiplied by the moment arm ( $\underline{\mathbf{r}} \times \hat{d}$ , perpendicular to  $d$ ); when the summation is made, most of the different  $d\tau$  will cancel except for masses not fitting into a spherically symmetric distribution (for which the net torque is zero). So it is  $\sim \sum m r^2 \times \nabla g \sin \theta \cos \theta$ , the two angular terms, one comes from taking component  $\parallel \frac{\partial}{\partial d}$ , another from  $\perp d$ ; The sum is over masses distribution *differing from* a spherical one, i.e. over the equatorial bulge.

Part of the reason for the rather big discrepancy (factor of 3) of the estimate with the actual observation is probably due to the fact that the equatorial bulge of the Earth is rather bigger than due to its *present* spinning rate. In the long history of the Earth, the spinning rate has slowed down a good fraction due to tiding force (c.f. question D.9). The ocean acts as a medium which drags behind, and over a long time the spinning angular momentum of the Earth is gradually transferred to the revolution of the moon around the Earth. A long time ago when the Earth was cooling down from semi-solid, it was spinning quite a bit faster.

**T.32 Tile.** Easier via finding the impulse  $P$  from the stick. This same impulse changes the linear motion, and make the tile flips around its principle axes.

Final linear velocity  $(0, 0, -v)$  given by:

$$P = mu - mv$$

The flipping (rotating around c.g.) rates:

$$\begin{aligned} \tau &= \begin{pmatrix} -bP \\ aP \\ 0 \end{pmatrix} = \underline{\underline{I}} \underline{\underline{\omega}} \\ &= \frac{m}{3} \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\ &= \frac{m}{3} \begin{pmatrix} b^2 \omega_x \\ a^2 \omega_y \\ (a^2 + b^2) \omega_z \end{pmatrix} \end{aligned}$$

This gives:  $\omega_x = -\frac{3P}{mb}$ ,  $\omega_y = -\frac{3P}{ma}$ ,  $\omega_z = 0$

We can then use energy conservation for elastic collision:

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2$$



Putting everything in terms of  $P$ , we eventually get  $P = \frac{2u}{7m}$ , and the rest of the result follows. The velocity at the corner is  $\underline{v} + \underline{\omega} \times \underline{r}$ , where  $\underline{r} = (-a, -b, 0)$  is the vector from the centre to the corner.

(Interestingly, the point of impact actually has its velocity totally reversed. This is probably true in general in elastic collisions.)

[Ans:  $\frac{5}{7}(0, 0, -u)$ ,  $\frac{2}{7}Mu(-b, a, 0)$ ,  $\frac{6}{7}u(-\frac{1}{b}, \frac{1}{a}, 0)$ ,  $(0, 0, +u)$ ]

### III. QUICKIES

**Q.1 Kettle.**  $E = C \cdot \Delta T$ ,  $C = 4\text{Jg}^{-1} \approx 4000 \text{ J Kg}^{-1}$ ,  $\Delta T \approx 100\text{K}$ .

Height:  $gh = C\Delta T$ , per mass,

$$\Rightarrow h = \frac{C\Delta T}{g} = 40\text{km}.$$

**Q.2 Liquid length.** Surface tension of water =  $0.07 \text{ N m}^{-1}$ , Latent heat is  $2.26 \times 10^6 \text{ J/Kg}$ , and density of water is  $10^3 \text{ Kg/m}^3$ . So the latent heat per volume is  $2 \times 10^9 \text{ J m}^{-3}$ .

The ratio is  $3 \times 10^{-11} \text{ m}$ , or  $0.3\text{\AA}$ .

This length is the length scale of (the inter-molecular force between) the molecules of water. Both latent heat and surface tension are to do with inter-molecular attractions.

**Q.3 50p.** Rolling on its edge: Instantaneous rotation about the opposite edge,  $T \approx \frac{1}{2}I'\dot{\theta}^2$ ,  $I' = I + ma^2 = \frac{1}{2}ma^2 + ma^2$  (parallel axis theorem), assuming small angle.

As the coin rotates, the centre of mass rises. The opposite tip has constant vertical height, so the centre moves as if it is suspended as a pendulum of length  $a$ .

$$V \approx mga(1 - \cos \theta) \approx mga \frac{\theta^2}{2}$$

So

$$E = T + V = \frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}mga\theta^2$$

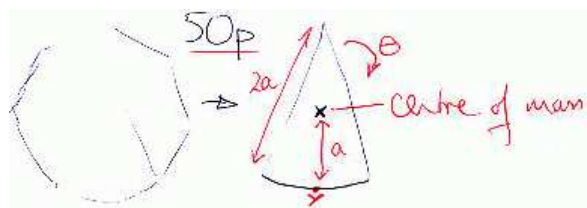
$$\Rightarrow I'\ddot{\theta} = -mga\theta$$

and therefore

$$\omega^2 = \frac{2g}{3a}$$

Since  $a \sim 0.01\text{m}$ , so  $\Rightarrow \omega^2 \approx 600$  or  $f = \frac{\omega}{2\pi} \approx 4 \text{ Hz}$ .

You can do various thing with it, but then there is only gravity  $g$ , mass  $m$ , and radius  $a$ , and the only combination with the right dimension is  $f \propto \sqrt{g/a}$ .

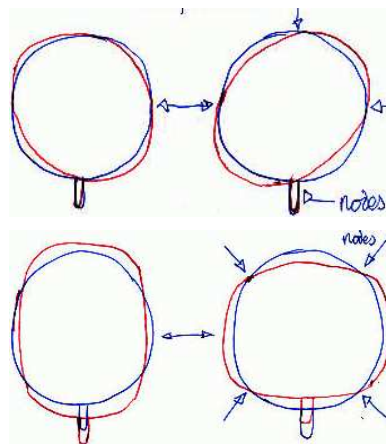


**Q.4 General relativity.**  $\theta \propto f(\frac{GM}{c^2d})$ . The angle  $\theta$  is dimensionless.

It might be argued that  $\theta \propto M$  on physical grounds, and it is true. Detailed general relativity argument gives  $\theta \approx \frac{4GM}{c^2d}$ . This involves solving the geodesic equation of the flight path of a massless object in the vicinity of the distortion of spacetime around a massive object, and is quite beyond the present scope.

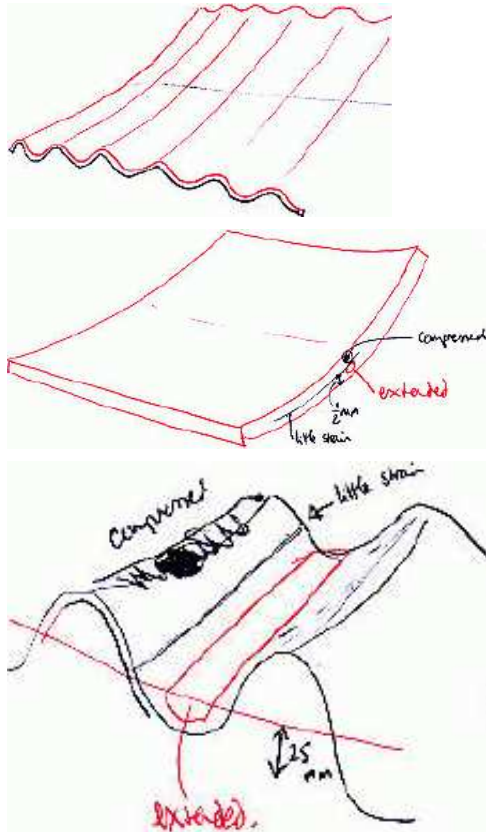
The effect can be verified by observing distant stars as the sun passes over the line of sight. Using the known estimates for the radius of the sun and the mass, the distortion as light passes over the surface of the sun is  $\sim 1.75''$ , a measurable effect ( $1''$  is 60th of a degree).

**Q.5 Cup.** If there were no handle, there are two lowest (fundamental) modes of a degenerate frequency. But with the handle, the degeneracy is removed and we have the two distinct modes:



The moving mass associated with the second is greater; thus it would have a lower frequency. Tapping at the nodes of these two would preferably excite the other mode, and giving a purer tone.

**Q.6 Corrugation** The strength to bending is proportional to the moment of the cross-section area about the centre. When it is corrugated, the same cross-section area is spread out from the central axis.

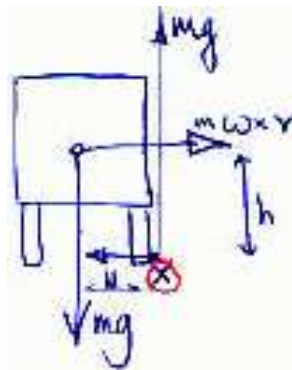


**Q.7 Train.** At the moment of toppling sideways, the reaction from the far wheels must be zero. Consider moment from the inner wheels:

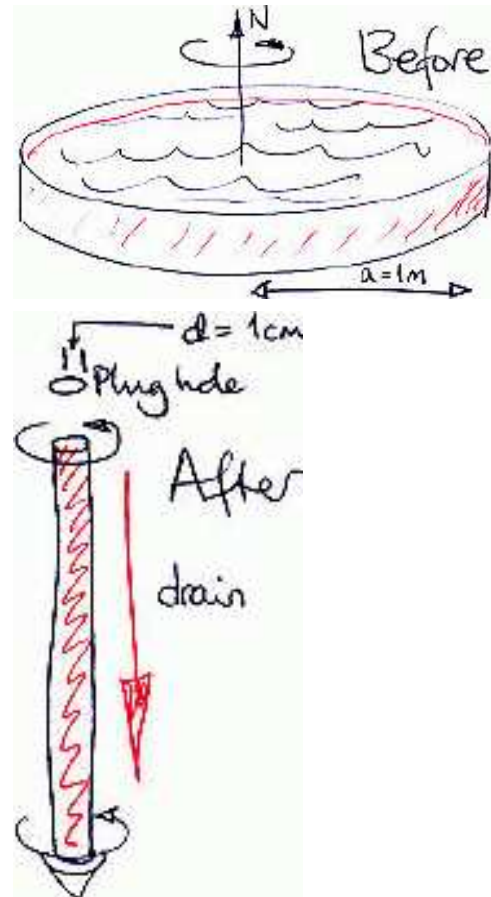
$$2\omega v h = mg \frac{w}{2}$$

Taking  $h \approx w$ ,  $v = \frac{g}{4\omega} = 36 \times 10^3 \text{ms}^{-1}$ , or  $72 \times 10^3 \text{mph}$ .

This velocity is higher than the orbital velocity ( $\sim 8 \text{km s}^{-1}, \sqrt{gR}$ ) or the escape velocity ( $\sim 11 \text{km s}^{-1}, \sqrt{2gR}$ ).



**Q.8 Bath.** The water spins faster as it drains into a narrow tube, because angular momentum roughly conserves in this situation of negligible friction, and the moment of inertia has greatly decreased.



$\omega = \frac{a^2}{b^2} \Omega$ ,  $\Omega$  the rate of rotation of the earth. Using reasonable numbers ( $a = 1 \text{m}$ ,  $b = 1 \text{cm}$ ) the rotation rate is about 1 rotation per 10 seconds.

In real life, random effects like the not-quite-vertical alignment of the tap from which the water came from, or a person playing around in the water, would give the water an initial (imperceptible) angular momentum. The rotation gets greatly magnified as it drains, due to the same effect above; the earth's effect is likely to be swamped and overshadowed.

E.g. a cup of coffee stirred has about the same magnitude of angular momentum as a bath tub due to earth's rotation.

It should be possible to argue in terms of Coriolis force? After all, the speeding up on going into the axis can possibly be expressed in acceleration by Coriolis force, and is of the same origin, just viewing in different frames.

**Q.9 Atmosphere.** 760mm of mercury or 32 feet of water exerts the same pressure as the atmosphere.

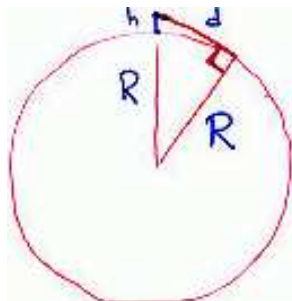
$$\rho_w g h_w = \rho_a g h_a, \text{ giving } h_a = 10 \text{ km.}$$

Nitrogen:

$$\frac{1}{2} m v_z^2 = \frac{1}{2} k T = m g h, \text{ gives } h_{\text{max}} \approx 5 \text{ km.}$$

These two are related; as the atmosphere is composed of nitrogen molecules that are (almost) in thermal equilibrium and the thickness of the atmosphere is as much as thermal agitation isn't sufficient to allow the molecules to escape earth's gravitation field.

**Q.10 Horizon.** It is really a geometry question, and 2nd part using result from Q9. You need the radius of earth.



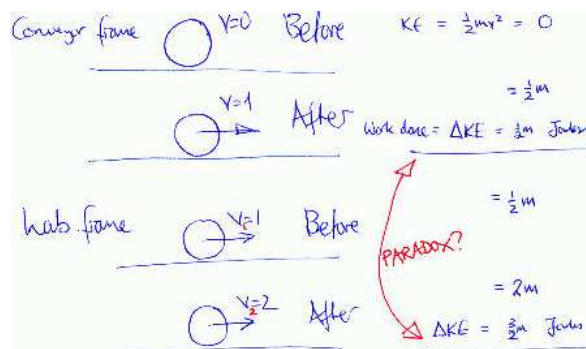
$$d^2 = (R+h)^2 - R^2 \approx 2Rh, \quad d = \sqrt{2Rh} \approx 4 \text{ km, for } h \sim 2\text{m, and } R = 6 \times 10^6 \text{ m.}$$

$$\text{Thickness of atmosphere} = \sqrt{2Rh_a} = 350 \text{ km.}$$

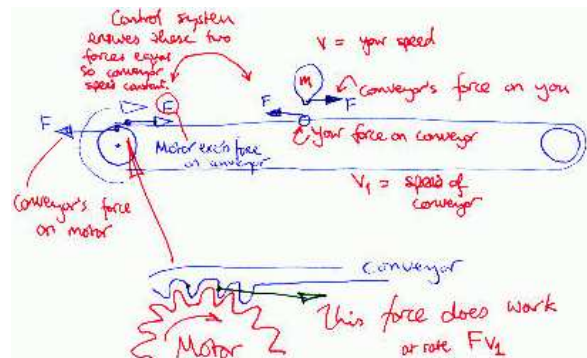
**Q.11 Music of the sphere.** Using dimensional analysis: acoustic velocity  $> 10^4 \text{ m/s}$  (any mechanical wave velocity would do), wave length  $\sim r = 6000\text{km} \leq 10^7\text{m}$  for the lowest normal mode. Very low  $\sim 10^{-3}$  Hertz. Earth quake is much higher in frequency as it involves surface modes rather than (in addition to?) body modes.

#### IV. DEEP THOUGHT

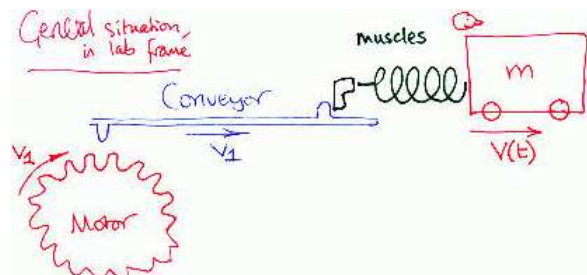
**D.1 Conveyor belt.** When you push yourself forward, you exert force on belt, which would have slowed down if not for the fact that the motor continues to drive it (and extra hard during your pushing). So the motor effectively do work on you.



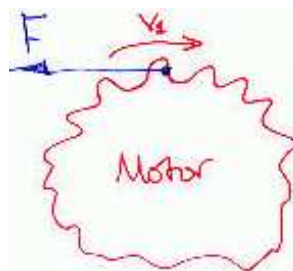
In the lab frame:



More details on the forces, etc:



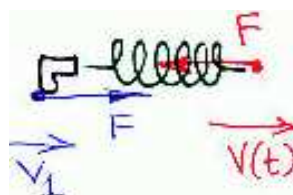
Assuming that force exerted on conveyor is  $F$ . Then the forces are as follows:



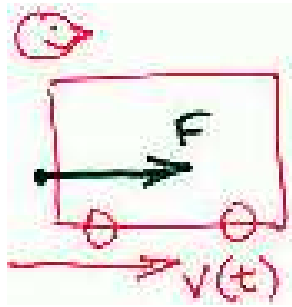
Work rate on motor =  $-Fv_1$  (i.e. motor is losing energy)



Workdone on conveyor =  $Fv_1 - Fv(t) = 0$  (i.e. conveyor has constant energy)



Work rate on muscle:  $= +Fv_1 - Fv(t) = -F(v(t) - v_1)$  (i.e. muscles are losing chemical energy)



Work rate on body and cart =  $F \cdot v(t)$ .

Total work done by these forces:

Work done on motor =  $\int (-Fv_1)dt$

Work done on conveyor = 0

Work done on muscles =  $\int (-F(v(t) - v_1))dt$

Work on body and cart =  $\int Fv(t)dt$

kinetic energy change of body and cart =  $\int Fv dt = \int mv \frac{dv}{dt} dt = \int mv dv = \frac{1}{2}mv_2^2 - \frac{1}{2}v_1^2$

This change comes from

(A) work done by the muscles =  $\int F(v - v_1)dt$

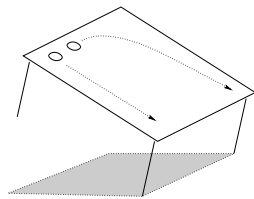
(B) and the work done by the motor =  $\int Fv_1 dt$

$\int Fv_1 dt = v_1 \int F dt = v_1 m \int \frac{dv}{dt} dt = v_1 m(v_2 - v_1)$

So (A) =  $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 - v_1 m(v_2 - v_1) = \frac{1}{2}m(v_2 - v_1)^2$

The work done by the motor (B) is the missing part.

**D.2 Shove ha'penny.** (a) The down-the-slope motion is the important one to concentrate on. Friction is parallel to instantaneous velocity, so the one that starts with a side-way shove will have a lesser friction component down the slope at the same elevation compared to the one that goes straight down. So the one going sideways arrives first.



(b) On the "up" trip, friction acts in the same sense as gravity; on the "down" trip, friction acts against gravity. Counting from the very top of the journey, they are the same distance up or down. It is zero velocity on one end, and motion are time-reversible except by with a different acceleration (due to friction). Therefore greater acceleration ("up") means shorter time.

A neater answer goes this way:

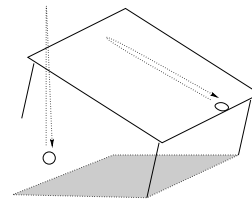
Consider the Kinetic Energy at a particular height on the way up and the way down:

Consider  $E_{\text{up}}(h)$  and  $E_{\text{down}}(h)$ :

Clearly  $E_{\text{up}} > E_{\text{down}}$

So  $v_{\text{up}}(h) > v_{\text{down}}(h)$

So the time to go up ( $= \int \frac{dh}{v_{\text{up}}}$ ) must be less than the time to come down.



(c) the horizontal motion makes it rather complicated; air resistance acts in the opposite sense as the instantaneous velocity and is also proportional to it. So the up trip experiences a bigger air resistance but it doesn't point quite in the vertical direction?

In the limit where air resistance dominates, the downwards trip probably has a bigger part of the journey with a vertical velocity close to critical (as limited by drag), since its horizontal momentum is gradually reduced over the whole journey. E.g. playing badminton: the feathery ball shoots forwards and up taking longer than the subsequent more vertical drop.

**D.3 European union.** There are on the whole equal and opposite number of cars going in opposite directions; and we are therefore asking how  $\uparrow\downarrow$  compared to  $\downarrow\uparrow$ . They differ in sense of rotation and therefore angular momentum. Fractional change in rotation rate = fraction change in angular momentum. Answer:  $10^{-22}$ . Very very tiny.

More details:

The whole earth conserves angular momentum, so if somebody cranks up a spinning top, he would be "pushing back" against the earth in the opposite sense, thus causing the earth to spin faster or slower.

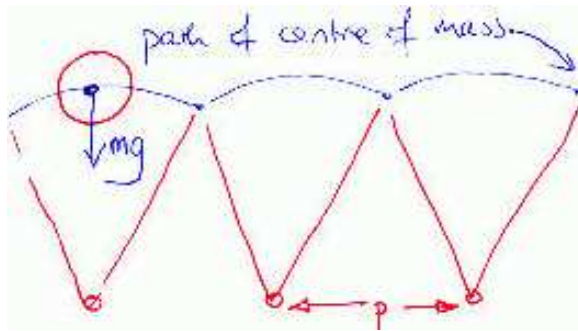
Change in angular momentum = (# of cars)  $mva$ , (# of cars)  $\approx$  a third of the population (say, one car per house hold of average three people) each spending an hour per day on the road.

$N \sim 10^6$ ,  $m \sim 10^3 \text{ kg}$ ,  $a \sim 10 \text{ m}$ ,  $v \sim 30 \text{ mph} \sim 10 \text{ ms}^{-1}$

The fractional change in the rotation rate is less than  $10^{-22}$ , or one nanosecond in 10,000 years.

**D.4 Walking.** The fastest comfortable walking is given by "toppling" oneself over one's legs. So the pacing frequency is about swinging oneself over one's legs, and  $f = \sqrt{g/l}$  steps per second. So bigger animal paces slower.





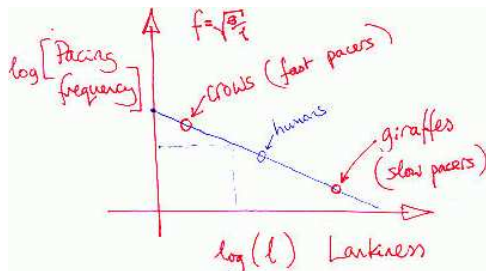
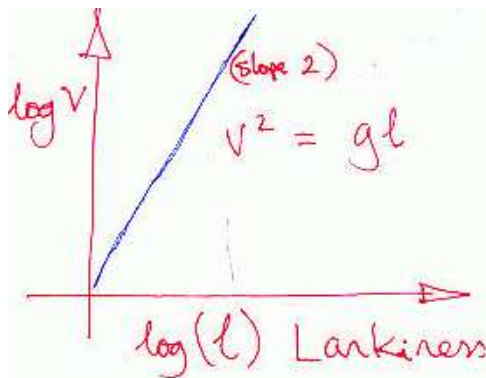
Walking involves pivoting like an inverted pendulum. If an inverted pendulum moves fast enough then the reaction force from ground drops to zero; any faster, you'll be flying.

This critical velocity is given by

$$m \frac{v^2}{l} = mg \Rightarrow v^2 = gl$$

Walking only involves a small drop in the centre of mass. Plausible pace length  $p$  is roughly  $l$ . At maximum speed, the pacing frequency  $f$  is roughly:

$$f = \frac{v}{p} = \sqrt{\frac{g}{l}} \text{ pace per second.}$$



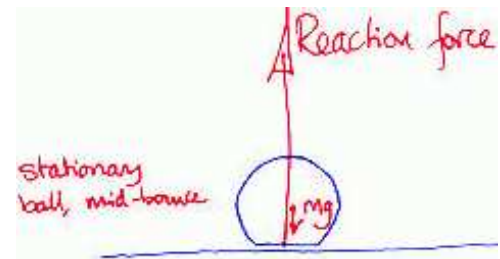
e.g. Although giraffes walk faster, they are slower pacers than crows, and human is somewhat in the middle.

For humans,  $l \sim 1\text{m}$ ,  $v = \pi\text{m s}^{-1}$ , or about 6 miles per hour. e.g. Girton College to Cambridge city centre is about half an hour's walk. Maximum pacing frequency  $\sim 3$  paces per second.

Note that the expressions for  $v$  and  $f$  could have been obtained by dimensional argument alone.

**D.5 Bouncing Balls.** The contact time can be approximately attributed to the transit time of a compressional pulse generated on impact. The pulse travels down to the other end and get reflected back due to the different impedance (between the body and air above) and becomes a rarefaction pulse which then pushes against the table and launches the ball up again, so it is of the order of  $\frac{\text{vertical size}}{\text{acoustic velocity}}$ .

If the floor is much softer, then the compressional acoustic pulse will be generated in the softer media instead and go to never-never-land and never come back.



The reaction force is much large than  $mg$ , since in mid-bounce, the ball reverses in direction and has a huge acceleration.

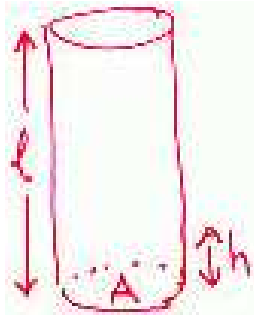
Treating the ball as a cube, we can use crude physics to estimate how big the forces are.

If the floor is rigid and the ball is compressible, one assumption to try is that all the kinetic energy of the ball is converted into elastic energy spreading through some part of the ball, then work out the displacement and strain that gives the necessary amount of elastic energy.



For a cylindrical rod, if we assume that it is all accelerating upwards then the strain must increase linearly towards the bottom, since we need a bigger stress to accelerate the whole mass.





Assuming equal acceleration  $a$  everywhere, as functions of  $h$ , the stress, strain, and energy stored per unit volume are:

$$\text{stress}(h) \cdot A = m \frac{l-h}{l} a$$

$$\text{strain}(h) = \frac{\text{stress}}{Y}$$

$$\text{energy}(h) = \frac{1}{2} \text{stress} \cdot \text{strain} = \frac{1}{2} \left[ m \left( \frac{l-h}{l} \right) \frac{a}{A} \right] \cdot \frac{1}{Y}$$

Now, if this integrated over the volume equal  $\frac{1}{2}mv^2$ , then we can solve for  $a$  and hence the stress and the strain.

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2Y} \int_0^l \left[ m \left( \frac{l-h}{l} \right) \frac{a}{A} \right] dh A \\ &= \frac{1}{2Y} \cdot \frac{1}{3} \left( \frac{ma}{A} \right)^2 l \cdot A \end{aligned}$$

When we put  $A = l^2$  (i.e. a cube), this simplifies to

$$ma = \sqrt{3}mv\sqrt{\frac{Y}{m}}\sqrt{l}$$

Ignoring the numerical factor  $\sqrt{3}$ , this says force is momentum per contact time  $\sqrt{\frac{m}{Y}}$ . Now, think of the speed of sound  $c$  as  $\sqrt{\frac{Y}{\rho}} = \sqrt{\frac{Yl^3}{m}}$ , the contact time is simply  $\frac{l}{c}$ .

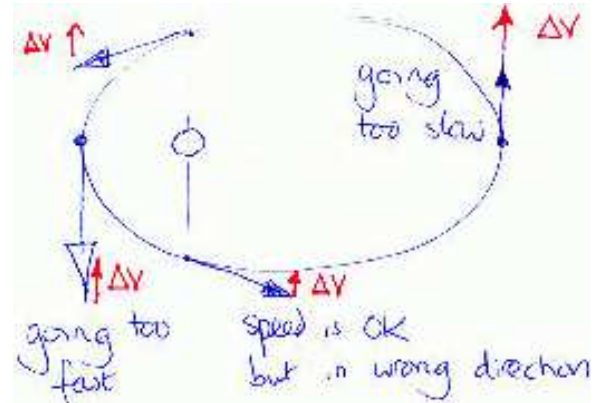
For a  $1 \text{ cm}^3$  steel ball dropped from 1m,  $c \sim 1.5 \times 10^4 \text{ m s}^{-1}$  (c.f.  $300 \text{ m s}^{-1}$  in air), contact time is  $\sim 10^{-6} \text{ s}$ . and maximum force  $\sim 3 \times 10^5 \times mg$ , its own weight, or, about  $20 \times 10^3 \text{ N}$ , or the weight of 20,000 apples. (10 apples per Kg...).

#### D.6 Burn time. ( $a$ , still in discussion with Dr MacKay)

In the extreme case, for a “free-fall” from infinity then come out right away (a very narrow ellipse), the velocity will be  $v^2 = 2GM/r$  (from energy) and mostly radial most of the time; while the require velocity to put it into circular orbit at any point is  $v^2 = GM/r$ , and totally tangential. It is then easy to see that the best place to do correction is at the far point. The near point will have a very large tangential velocity, but the required velocity is also large and very different; so the difference is likely to be large as well.

Maybe it isn’t as simple as this. Somebody needs to do a numerical simulation to work out the details.

A burn generates a change in the momentum. So the answer to this question involves figuring out what the momentum in a circular orbit would be, and seeing when in the orbit  $\Delta p$  (between the current  $p$  and the required  $p$  is smallest.



At the near point, the required change is to *increase* the velocity, while, at the far point, the required change is to *decrease* the velocity; somewhere intermediate the magnitude is right but the direction is wrong.

*Dr MacKay's note:* Steve Gull (previous lecturer?) says it makes little difference, but the intermediate positions are best. (quoted verbatim without proof).

(b) The thruster gives a  $\Delta v$ , independent of what  $v$  is. To leave orbit, want highest total energy  $E$  and to have  $E > 0$ . Thruster modifies the current kinetic energy; fixed  $\Delta v$ , and changing  $(v + \Delta v)^2$ , so should do it when  $v$  is biggest; i.e. at the closest point to earth.

**D.7 Cornering.** The “falling over” has a small component in the direction of motion (since the bike is turning into the direction of “fall over”).

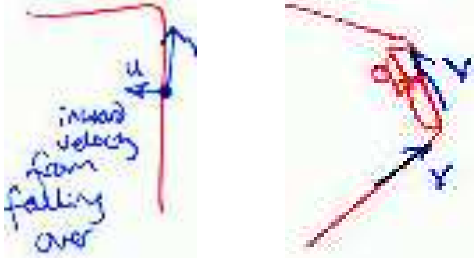
- You fall over like a compound pendulum.



- You’re now speeding to the left, with the wheel still going ahead.



- Turn left, still leaning over; you've now added together  $u$  and rotated  $v$ , so your speed is increased.



### D.8 Bad Working.

- The potential is  $-\frac{GM}{r}$ , not  $\frac{GM}{r}$ .
- We can ignore the external shell only if we are comparing the *force*, but we can't ignore its contribution to the potential.



- If we work out the total potential, it is:  

$$-\frac{Gm_i}{r} + \int_{r'=r}^R -\frac{Gm_e(r')}{r'} dr'$$
 where  $m_i$  = internal mass,  $m_e(r')$  = external mass.  
 But that's a pain, so it's easier simply to find the force directly.

$$F(r) = -\frac{4\pi}{3} \frac{G\rho r^3}{r^2} = -\frac{4\pi}{3} G\rho r$$

So we have simple harmonic motion with

$$\omega^2 = \frac{4}{3} \pi G\rho$$

If you have to go the potential route,

$$\begin{aligned} V &= -\frac{Gm_i}{r} - \int_{r'=r}^R 4\pi\rho Gr' dr' \\ &= -\frac{4\pi}{3} G\rho r^2 - \frac{4\pi}{2} \rho G(R^2 - r^2) \\ &= -2\pi\rho GR^2 + \frac{2\pi}{3} \rho Gr^2 \end{aligned}$$

The first term being constant, and

$$F = \frac{\partial V}{\partial r} = -\frac{4\pi}{3} \rho Gr$$

indeed, the expected answer.

**D.9 Tides.** The quick answer is that, because the gravitation potential is convex upward, the attraction to the bit of near water is stronger than linear, while the bit of far water is weaker than linear. And the centrifugal/centripetal force is  $mr\omega^2$ , i.e. linear. Therefore in the near side to the sun, gravitational pull is slightly stronger than to keep the water in orbit, while in the far side the gravitation pull is weaker than is required to keep the water in orbit; so on either end, the water level are slightly deeper.

Moon is closer, so it has a slightly bigger influence but not by much. Once in a while you get a near-cancellation of moon/sun influence, and “miss” one tide. (therefore only get the other ones before and after; twice a month?).

For a much simplified argument in 1-dimension (full treatment further below), let's just concentrate on the force, and on the radial direction through the centre of the Earth—Sun sytem. The required centripetal force for orbiting motion is  $mr\omega^2$  at a particular distance  $r$  from the sun, for orbiting rate  $\omega$ . The attraction from the sun is  $\frac{GMm}{r^2}$  from the sun. At the position of the Earth's centre, they must be the same:

$$\frac{GMm}{r^2} = mr\omega^2$$

But on either end on the surface of the earth, the difference is:  $\frac{GMm}{(r \pm R)^2} - m(r \pm R)\omega^2 \approx \frac{GMm}{r^2} [1 \mp \frac{2R}{r}] - m(r \pm R)\omega^2 = \mp mR (\frac{2GM}{r^3} + \omega^2)$  This expression is -ve (i.e. attraction too weak compared to required centripetal force) away from the sun (+ve  $R$ ), and +ve (i.e. attraction too strong) in the near side (-ve  $R$ ).

Viewing from the Earth's (rotating) frame, there is effectively a tiny force on either end of magnitude  $|mR (\frac{2GM}{r^3} + \omega^2)| = 3m\omega^2 R$  in addition to Earth's gravitation pull, pulling things away from the centre.

So any loose objects (e.g. water...) on the surface of the earth would redistribute accordingly, until it becomes an equipotential surface. E.g. For the near side, since the gravitation pull due to the sun is slightly too strong to keep the water in the same orbiting period, there would be an accumulation of water. Similar argument for the far end.

The estimation of the tide height has to be done via the potential energy as in the end of the energy argument (read on). The water level is an equipotential surface, so the potential of water perpendicular to the Earth—Sun line must be the same as that of along the line on either end:

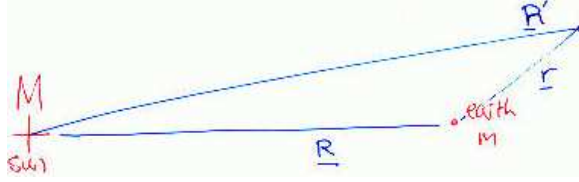
$$-\frac{GM_e}{R} = -\frac{GM_e}{R + \delta R} - \frac{1}{2} 3\omega^2 (R + \delta R)^2$$

$$\text{Or, } \frac{GM_e}{R^2} \delta R = g\delta R \approx \frac{3\omega^2}{2} R^2.$$

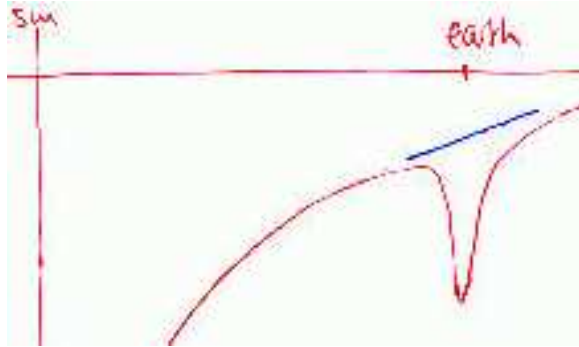
The most elegant way to describe tides is in terms of potentials rather than forces.

The potential due to the earth and the sun, for example, is given by:

$$V = -\frac{GM}{|\underline{\mathbf{R}}'|} - \frac{Gm}{|\underline{\mathbf{r}}|}$$



The potential along a line passing close to the earth is as follows:



Notice that near the earth, the sun's contribution is roughly linear. Tides come from the second-order corrections to this leading-order term.

Let's do the sun alone first.

Because the sun-earth system is rotating, the earth and its oceans are accelerating towards the sun. This acceleration can be described by a fictitious potential, [A potential whose (-ve) derivative is the centrifugal force]:

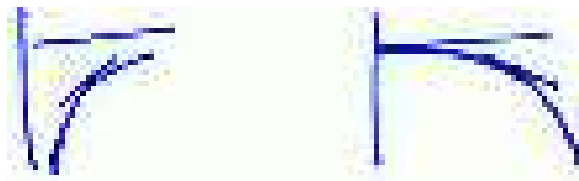
$$V_{\text{fict}}(\underline{\mathbf{R}}') = -\frac{1}{2}\omega^2|\underline{\mathbf{R}}'|^2$$

per unit mass. [This potential doesn't produce Coriolis forces, but don't worry, everything is "stationary"]

The total potential is thus:

$$V_{\text{eff}}(\underline{\mathbf{r}}) = -\frac{GM}{|\underline{\mathbf{R}}'|} - \frac{GM}{|\underline{\mathbf{r}}|} - \frac{1}{2}\omega^2|\underline{\mathbf{R}}'|^2$$

Now, at the centre of the earth, the derivative of  $-\frac{GM}{|\underline{\mathbf{R}}'|}$  exactly balances the derivative of  $\frac{1}{2}\omega^2|\underline{\mathbf{R}}'|^2$ :



Consider the Taylor expansion of  $V_{\text{eff}}(\underline{\mathbf{r}})$ :

$$V_{\text{eff}}(\underline{\mathbf{r}}) = -\frac{GM}{|\underline{\mathbf{R}}'|} + \text{const.} + \underline{\mathbf{a}} \cdot \underline{\mathbf{r}} + \frac{1}{2}\underline{\mathbf{r}}^T \underline{\mathbf{B}} \underline{\mathbf{r}} + \dots$$

Where

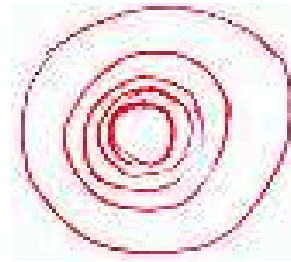
$$\underline{\mathbf{a}} = \frac{\partial}{\partial \underline{\mathbf{r}}} \left[ -\frac{GM}{|\underline{\mathbf{R}} + \underline{\mathbf{r}}|} - \frac{1}{2}\omega^2|\underline{\mathbf{R}} + \underline{\mathbf{r}}|^2 \right] \bigg|_{\underline{\mathbf{r}}=0} = 0$$

and

$$\underline{\mathbf{B}} = \frac{\partial^2}{\partial \underline{\mathbf{r}} \partial \underline{\mathbf{r}}} \left[ -\frac{GM}{|\underline{\mathbf{R}} + \underline{\mathbf{r}}|} - \frac{1}{2}\omega^2|\underline{\mathbf{R}} + \underline{\mathbf{r}}|^2 \right] \bigg|_{\underline{\mathbf{r}}=0}$$

The water covering the earth fills up an equipotential (at least, it would if there were no land masses rushing by).

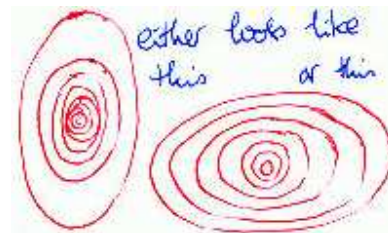
The equipotentials from the  $-\frac{GM}{|\underline{\mathbf{r}}|}$  term look like perfect circles:



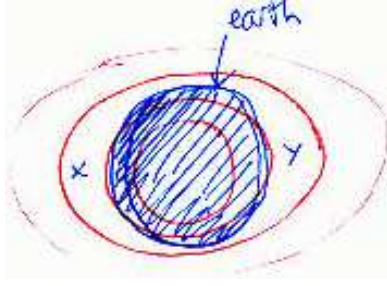
The second term  $\underline{\mathbf{a}} \cdot \underline{\mathbf{r}}$  is zero. The third term  $\frac{1}{2}\underline{\mathbf{r}}^T \underline{\mathbf{B}} \underline{\mathbf{r}}$  can only have the general form of an ellipsoid: in fact, since it is uniaxial (the direction of the sun), it can only be one of 4 forms: (a) prolate ellipsoid (e.g. baseball shape, thick cigar), (b) oblate ellipsoid (i.e. a flatten sphere), (c) planar,  $\perp$  to the axis, (d) axial.

[What this means is that in the frame where  $\underline{\mathbf{B}}$  is diagonalized,  $\frac{1}{2}\underline{\mathbf{r}}^T \underline{\mathbf{B}} \underline{\mathbf{r}}$  is of the form  $\propto (px^2 + qy^2 + rz^2)$ , uniaxial means  $p = q$ , and since the  $\nabla^2 V$  for gravitational field is always negative,  $(p+q+r) < 0$ , and for simple geometries like what we have here,  $p, q, r$  are each  $\leq 0$  (hence ellipsoids, for  $p, q, r$  all  $< 0$ ). Cases (c) and (d) corresponds to the case when either  $p$  or  $r$  are accidentally zero.]

In fact, in this case  $\frac{1}{2}\underline{\mathbf{r}}^T \underline{\mathbf{B}} \underline{\mathbf{r}}$  looks like case (c). But to answer the question, "why are there two high tides", it doesn't matter. Any form of perturbation  $\frac{1}{2}\underline{\mathbf{r}}^T \underline{\mathbf{B}} \underline{\mathbf{r}}$  (except for the special case of  $\underline{\mathbf{B}} = \underline{\mathbf{I}}$ ) will produce equipotentials (when added to the spherical central potential from the earth to itself) that are ellipsoids.



Either way, if you plopp a spherical earth at the centre of these contours,

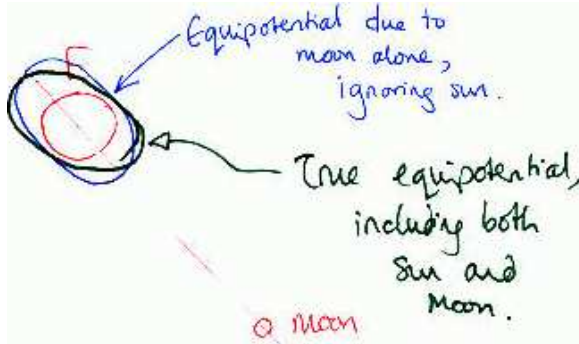


it must be that there's two places  $x, y$  where the equipotential is deepest, so two high tides per solar day since the earth spins around once per day.

As for the influence of the moon: For the Earth-moon system, the same story applies, with the potential being the sum of the sun potential that we just discussed, plus a centrifugal potential from the rotation about  $C$ , the centre of rotation of the Earth-moon system. (Actually it may be inside the earth, but it's off-centre, that's all that matters), plus a potential  $-\frac{GM_{\text{moon}}}{|\mathbf{r}'_{\text{moon}}|}$ .

Just as with the sun, these two potentials cancel to leading order, and to second order they produce a hump-like  $\frac{1}{2}\mathbf{r}^T \underline{\underline{\mathbf{B}}} \mathbf{r}$  term oriented along the earth-moon direction.

The sum of the sun's  $\frac{1}{2}\mathbf{r}^T \underline{\underline{\mathbf{B}}}_{\text{sun}} \mathbf{r}$  and the moon's  $\frac{1}{2}\mathbf{r}^T \underline{\underline{\mathbf{B}}}_{\text{moon}} \mathbf{r}$  perturbation, and  $-\frac{GM}{|r|}$  is still a total potential whose equipotentials are ellipses.



[Mathematically, it is just diagonalized to different axes.]

As we'll now confirm, the moon's  $\frac{1}{2}\mathbf{r}^T \underline{\underline{\mathbf{B}}} \mathbf{r}$  is bigger than the sun's, so the high tides, mid-ocean, correlate with the moon's direction.

If Sun  $\perp$  to moon, the equipotentials are more spherical — “Neap tides”. If sun  $\parallel$  to moon, then the equipotentials are more ellipsoidal — “Spring tides”.

Back to the algebraic manipulation.

$$\underline{\underline{\mathbf{a}}} = \frac{\partial}{\partial \mathbf{r}} \left[ -\frac{GM}{|\mathbf{R} + \mathbf{r}|} - \frac{1}{2}\omega^2 |\mathbf{R} + \mathbf{r}|^2 \right]$$

$$= \frac{\partial}{\partial r_i} \left[ -\frac{GM}{\sqrt{(R_j + r_j)(R_j + r_j)}} - \frac{1}{2}\omega^2 (R_j + r_j)^2 \right]$$

$$= +\frac{GM}{\{(R_j + r_j)^2\}^{3/2}} (R_i + r_i) - \omega^2 (R_i + r_i)$$

$$\text{Verifying at } r_i = 0, \frac{\partial}{\partial r_i} V = +\frac{GM}{\{R_j^2\}^{3/2}} R_i + \omega^2 R_i = \left[ \omega^2 - \frac{GM}{|R|^3} \right] R_i = 0.$$

Now let's get  $\underline{\underline{\mathbf{B}}}$ :

$$B_{ki} \propto \frac{\partial^2}{\partial r_k \partial r_i} [\dots] \Big|_{\mathbf{r}=0}$$

$$= \frac{\partial}{\partial r_k} (R_i + r_i) \left[ -\omega^2 + \frac{GM}{\{(R_j + r_j)^2\}^{3/2}} \right] \Big|_{\mathbf{r}=0}$$

$$= \delta_{ik} \left[ -\omega^2 + \frac{GM}{\{(R_j + r_j)^2\}^{3/2}} \right] \Big|_{\mathbf{r}=0}$$

$$+ (R_i + r_i) \frac{\partial}{\partial r_k} \left[ +\frac{GM}{\{(R_j + r_j)^2\}^{3/2}} \right] \Big|_{\mathbf{r}=0}$$

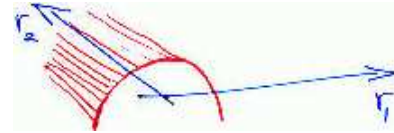
$$= \delta_{ik} \cdot 0$$

$$+ (R_i + r_i) \left( \frac{3}{2} \right) \frac{-GM}{\{(R_j + r_j)^2\}^{5/2}} \cdot 2(R_k + r_k)$$

$$= -3 \frac{GM}{|R|^5} R_i R_k$$

$$\text{or } = -3 \frac{GM}{|R|^3} \hat{R}_i \hat{R}_k, \quad \hat{R}_i \text{ unit vector in } R_i \text{'s dir'}$$

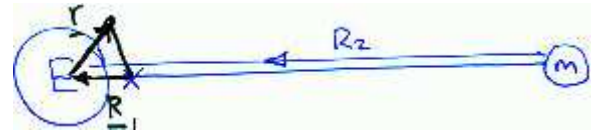
This is a hump shaped function:



Using  $\omega^2 = \frac{GM}{|R|^3}$ , we can rewrite  $B_{ik}$  as:  $B_{ik} = -3\omega^2 \hat{R}_i \hat{R}_k$ .

Check dimensions:  $[\frac{1}{2}\mathbf{r}^T \underline{\underline{\mathbf{B}}} \mathbf{r}]$  = potential per unit mass =  $[E M^{-1}]$ ;  $[\underline{\underline{\mathbf{B}}}] = [E M^{-1} L^{-2}] = [T^{-2}]$ , correct.

For the earth-moon system, we need to worry about the centre of mass and all that.



Centrifugal term is  $\sim (\mathbf{R}_1 + \mathbf{r})^2$ , and moon gravity term is  $\sim \frac{1}{|\mathbf{R}_2 + \mathbf{r}|}$ , so when we differentiate, we need to keep track of which  $\mathbf{R}$ 's is which.

Re-doing the previous,

$$\begin{aligned} \frac{\partial}{\partial r_i} \left[ -\frac{GM}{\sqrt{(R_j^{(2)} + r_j)(R_j^{(2)} + r_j)}} - \frac{1}{2}\omega^2(R_j^{(1)} + r_j)^2 \right] \\ = +\frac{GM}{\{(R_j^{(2)} + r_j)^2\}^{3/2}}(R_i^{(2)} + r_i) - \omega^2(R_i^{(1)} + r_i) \\ = 0, \quad \text{if } \frac{GM}{|R_2|^3}R_i^{(2)} = \omega^2 R_i^{(1)} \end{aligned}$$

Not  $\omega^2 = \frac{GM}{R^3}$ , but  $\omega^2 = \frac{GM}{R_2^3} \frac{|R_2|}{|R_1|}$ , and  $\frac{|R_2|}{|R_1|} \approx \frac{m_e}{m_m} \approx 100$ .

So then when we find  $B_{ik} = -3 \frac{GM}{|R_2|^2} \hat{R}_{2,i} \hat{R}_{2,k}$ , this can be rewritten as  $B_{ik} = -3\omega^2 \frac{|R_1|}{|R_2|} \approx -3 \frac{\omega^2}{100}$ .

Thus  $(\underline{\underline{B}}_{\text{sun}})_{ik} = -3\omega_s^2 \hat{R}_{s,i} \hat{R}_{s,k}$ , and  $(\underline{\underline{B}}_{\text{moon}})_{ik} = -3 \frac{\omega_m^2}{100} \hat{R}_{m,i} \hat{R}_{m,k}$

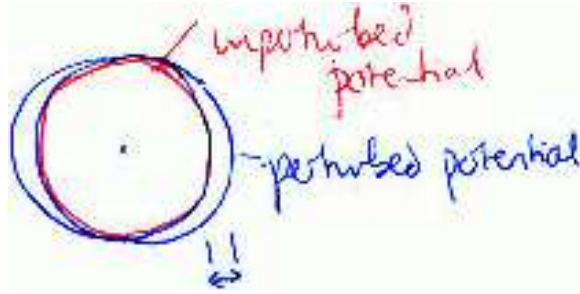
So the ratio of these two perturbation is:

$$\frac{B_m}{B_s} = \frac{\omega_m^2}{\omega_s^2} \frac{1}{100} \approx \frac{144}{100}$$

(assuming 12 lunar months per year) Actually there's 13.4 moons per year, so  $\frac{B_m}{B_s} \approx 1.7$ .

The height of the tide is can be obtained by comparing the leading term  $\frac{GM_e}{|\underline{\underline{r}}|}$  with the perturbation,

$$\left[ -\frac{3\omega^2}{100} = \underline{\underline{B}} \right], \quad \frac{1}{2} \underline{\underline{r}}^T \underline{\underline{B}} \underline{\underline{r}} :$$



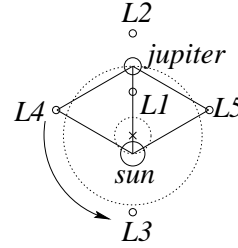
$$\begin{aligned} h &\approx \frac{|\Delta V|}{\left| \frac{\partial V_{\text{unperturbed}}}{\partial r} \right|} \\ &= \frac{\left\| \frac{1}{2} \underline{\underline{r}}^T \underline{\underline{B}} \underline{\underline{r}} \right\|_{\text{on surface}}}{\left| \frac{GM_e}{r^2} \right|} \\ &= \frac{\frac{1}{2} \frac{3\omega^2}{100} (r_e)^2}{g} \\ &= \underline{0.24m}, \quad \text{Using } \omega = 2\pi \text{ per month} \end{aligned}$$

Coastal tides are much bigger because the waves generated mid-ocean get amplified as they go up the continental shelf because the wave velocity depends on depth.

Honolulu tides are about one foot in range; this is perfectly with our 24cm estimate.

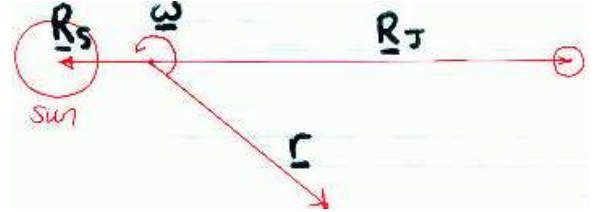
[Ans: Tidal range in Honolulu is of order 1 foot.]

**D.10 Lagrange points.** The explicit solution for L1, L2 and L3 is difficult since it turns out to involve a worse-than-cubic equation. As for L4 and L5 it's quite easy to confirm they are fixed points, but finding whether they are stable is a much harder problem.



A point in a rotating frame is a fixed point if it's a stationary part of the effective potential, per unit mass:

$$V_{\text{eff}}(\underline{\underline{r}}) = \frac{GM_s}{|\underline{\underline{r}} - \underline{\underline{R}}_s|} - \frac{GM_J}{|\underline{\underline{r}} - \underline{\underline{R}}_J|} + V_{\text{fict}}(\underline{\underline{r}})$$



Where  $V_{\text{fict}}(\underline{\underline{r}})$  is the potential responsible for the centrifugal force:  $\underline{\underline{f}}_c = -\underline{\underline{\omega}} \times (\underline{\underline{\omega}} \times \underline{\underline{r}}) = (\omega^2 - \underline{\underline{\omega}} \underline{\underline{\omega}}^T) \underline{\underline{r}}$ .

$$V_{\text{fict}}(\underline{\underline{r}}) = -\frac{1}{2} \underline{\underline{r}}^T (\omega^2 - \underline{\underline{\omega}} \underline{\underline{\omega}}^T) \underline{\underline{r}}$$

Now,  $\underline{\underline{R}}_s$  and  $\underline{\underline{R}}_J$  are parallel and  $M_s \underline{\underline{R}}_s + M_J \underline{\underline{R}}_J = 0$ .

[Note that if a particle is moving then additional Coriolis forces act that are not mentioned in  $\frac{\partial}{\partial \underline{\underline{r}}} V_{\text{eff}}(\underline{\underline{r}})$ , so we can't determine stability from  $V_{\text{eff}}$ .

For equilibrium, we differentiate  $V_{\text{eff}}(\underline{\underline{r}})$  (with summation convention):

$$\begin{aligned} \frac{\partial V_{\text{eff}}(\underline{\underline{r}})}{\partial r_i} &= \sum_{\alpha=s,J} \frac{GM_\alpha (r_i - R_i^{(\alpha)})}{[(r_j - R_j^{(\alpha)})^2]^{3/2}} - [\omega^2 \delta_{ij} - \omega_i \omega_j] r_j \\ &= 0 \end{aligned}$$

Rearranging, we have:

$$\sum_{\alpha=s,J} \frac{GM_\alpha \underline{\underline{R}}^{(\alpha)}}{|\underline{\underline{r}} - \underline{\underline{R}}_\alpha|^3} = \left[ \sum_{\alpha=s,J} \frac{GM_\alpha}{|\underline{\underline{r}} - \underline{\underline{R}}_\alpha|^3} - \omega^2 \right] \underline{\underline{r}} + \underline{\underline{\omega}} (\underline{\underline{\omega}}^T \underline{\underline{r}})$$

This is the condition for a fixed point. The left hand side is a pure multiple of  $\underline{\underline{R}}_s$  and  $\underline{\underline{R}}_J$  which are parallel.



The right hand side is a sum of a multiple of  $\underline{\mathbf{r}}$  and  $(\underline{\omega}^T \underline{\mathbf{r}})$  times  $\underline{\omega}$ .

There is no way that this can have any component in direction of  $\underline{\omega}$ , so we must have  $(\underline{\omega}^T \underline{\mathbf{r}}) = 0$ .

So our fixed point equation is:

$$\left( \sum_{\alpha} \dots \right) \underline{\mathbf{R}}^{(\alpha)} = [\dots] \underline{\mathbf{r}}$$

Call  $(\sum_{\alpha} \dots) = \mathbf{A}$  and  $[\dots] = \mathbf{B}$ , the fixed point equation can only be satisfied if either both  $\mathbf{A}$  and  $\mathbf{B}$  are zero [L4, L5], or if they are non-zero,  $\underline{\mathbf{r}} \parallel \underline{\mathbf{R}}_{s,J}$  [L1, L2, L3].

If  $A = 0$  and  $B = 0$ ,

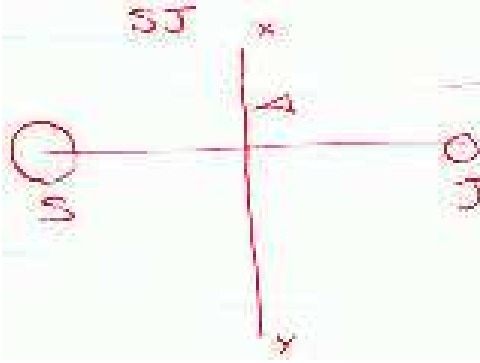
**A = 0:**

$$\frac{GM_s}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_s|^3} |\underline{\mathbf{R}}_s| = \frac{GM_J}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_J|^3} |\underline{\mathbf{R}}_J|$$

using  $M_s |\underline{\mathbf{R}}_s| = |M_J \underline{\mathbf{R}}_J|$ , we get

$$|\underline{\mathbf{r}} - \underline{\mathbf{R}}_s|^3 = |\underline{\mathbf{r}} - \underline{\mathbf{R}}_J|^3$$

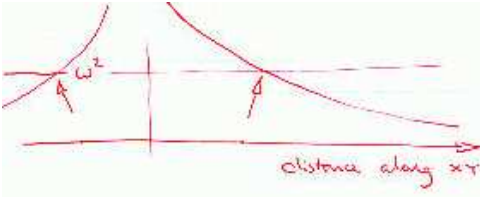
i.e.  $\underline{\mathbf{r}}$  must be on the place bisecting  $s-J$



**B = 0:**

$$\frac{GM_s}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_s|^3} + \frac{GM_J}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_J|^3} = \omega^2$$

This equation has two symmetrical solutions.



Because  $s$  and  $J$  are in circular orbits, around the origin,

$$\frac{GM_s}{|\underline{\mathbf{R}}_s - \underline{\mathbf{R}}_J|^2} = \omega^2 |\underline{\mathbf{R}}_J|, \frac{GM_J}{|\underline{\mathbf{R}}_s - \underline{\mathbf{R}}_J|^2} = \omega^2 |\underline{\mathbf{R}}_s|$$

So we can confirm that the two solutions are given by L4 and L5, which are defined by:

$$\underline{\mathbf{r}}^T \underline{\omega} = 0 \text{ \& } |\underline{\mathbf{r}} - \underline{\mathbf{R}}_s| = |\underline{\mathbf{r}} - \underline{\mathbf{R}}_J| = |\underline{\mathbf{R}}_J - \underline{\mathbf{R}}_s|,$$

By substituting:

$$\frac{GM_s}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_s|^3} + \frac{GM_J}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_J|^3} = \frac{\omega^2 |\underline{\mathbf{R}}_s|}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_s|} + \frac{\omega^2 |\underline{\mathbf{R}}_J|}{|\underline{\mathbf{r}} - \underline{\mathbf{R}}_J|} = \omega^2$$

For L1, L2, L3, if  $\underline{\mathbf{r}}$  is parallel to  $\underline{\mathbf{R}}_s$  and  $\underline{\mathbf{R}}_J$ , then we have a scalar problem to solve.

Let  $\beta M_s = M_J$  and  $\underline{\mathbf{R}}_s = -\beta \underline{\mathbf{R}}_J$ ,

$$\omega^2 = \frac{GM_s}{(R_s - R_J)^2 R_J} = \frac{GM_J / \beta}{(1 + \beta)^2 R_J^3}$$

Let  $\underline{\mathbf{r}} = \alpha \underline{\mathbf{R}}_J$ , we want to find  $\alpha$ .

The fixed-point equation gives,

$$\begin{aligned} \frac{GM_s}{|\alpha + \beta|^3} \frac{-\beta R_J}{R_J^3} + \frac{GM_J}{|\alpha - 1|^3} \frac{R_J}{R_J^3} \\ = \left[ \frac{GM_J}{|\alpha + \beta|^3 R_J^3} + \frac{GM_J}{|\alpha - 1|^3 R_J^3} - \omega^2 \right] \alpha R_J \end{aligned}$$

$$\frac{-\beta}{|\alpha + \beta|^3} + \frac{\beta}{|\alpha - 1|^3} = \alpha \left[ \frac{1}{|\alpha + \beta|^3} + \frac{\beta}{|\alpha - 1|^3} - \frac{1}{|\beta + 1|^2} \right]$$

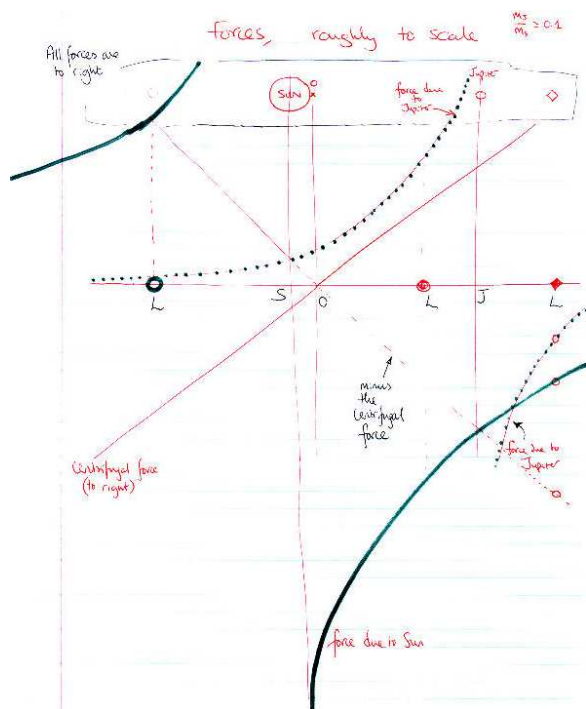
This looks like it'll give a cubic equation?

Multiply by  $|\alpha + \beta|^3 |\alpha - 1|^3$ ,

$$\begin{aligned} \beta [|\alpha - 1|^3 - |\alpha + \beta|^3] \\ + \alpha \left[ |\alpha - 1|^3 + \beta |\alpha + \beta|^3 - \frac{|\alpha + \beta|^3 |\alpha - 1|^3}{|\beta + 1|^2} \right] = 0 \end{aligned}$$

Highest degree term is  $\alpha^7$ , but from qualitative arguments we only expect three points. The absolute signs probably rule out more than a few roots. We could pin them down by perturbation methods, assuming  $M_J \ll M_s$





No, it is actually only a 5th degree equation. If we look back to the original equilibrium condition, written in scalar form, is simply force from sun ( $\propto |\alpha + \beta|^{-2}$ ) minus force from Jupiter ( $\propto |\alpha - 1|^{-2}$ ) equal centrifugal force ( $\propto \alpha$ ). The above complicated 7th-degree equation is reducible to :

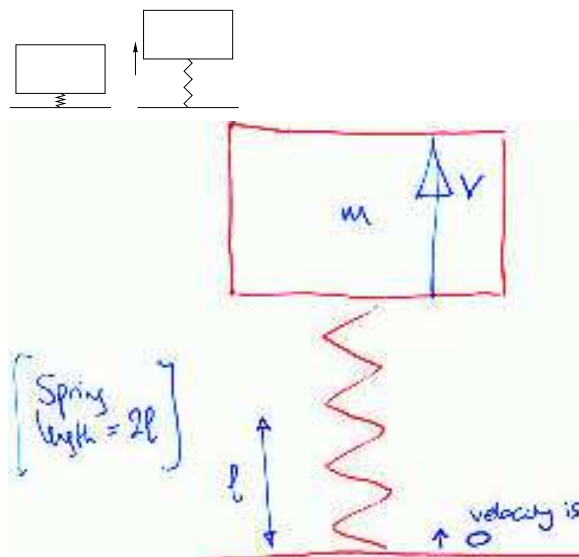
$$\frac{1}{|\alpha + \beta|^2} - \frac{\beta}{|\alpha - 1|^2} = \frac{1}{|1 + \beta|^2} \alpha$$

There are probably two imaginary roots of  $\alpha$ 's corresponding to negative  $|\alpha + \beta|^2$  and  $|\alpha - 1|^2$ . In any case, on examining physically the trends of the forces (both decreasing from their stars) and the required centrifugal force (increasing from the c.g.), it is obvious that there can only be one and only one solution in each of the 3 regions of between the stars or beyond either sides.

Guess work: We were given that L4, L5 and L1 are actually currently occupied so they must be stable positions. (This is really cheating...). L2 and L3 are unstable. We can reason as follows: the required centripetal force of a stationary satellite is linear from the c.g. of the system. But if the satellite is displaced slightly towards the centre, it should be apparent that the attraction force to the closer gravitating object (sun or Jupiter) increases much faster than linear and hence it will simply accelerate away from L2 and L3 and fall into the stars. On the other hand, a satellite at L1 may be pushed back to it if it is slightly displaced, because the two stars are on either side and the attraction forces partially cancel.

**D.11 Zebedee.** At the moment of separation, top of spring will travel with velocity  $v$ , same as the mass,

while the bottom is just off ground with velocity zero; and the whole lot then goes with mean velocity  $v/2$ . So the height eventually attained is only a quarter compared to the mass. The spring also does a bit of stretching and contracting during flight.



At the moment when spring stops pressing on  $m$ , its top end is going at  $v$ , like  $m$ , and its bottom end is going at 0. The spring is uniform, so its centre of mass is going at  $v/2$ .

The spring goes to a height  $h/4$  where  $h = \frac{v^2}{2g}$  is the height the block goes to.

[Strictly, the spring goes a height  $L$  less than this because its centre of mass starts off  $l$  lower, as shown.]

Since the top and bottom start at different speeds, the spring oscillates to and fro as it flies.

**D.12 Car areas.** gallon is volume. So the dimension is a cross-section area; the size of a tube of petrol that the car would suck up continuously for refuelling while in motion.

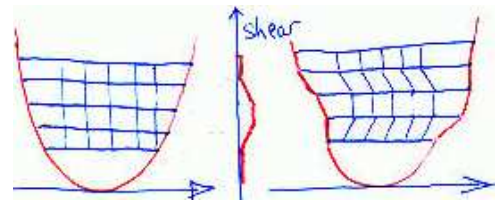
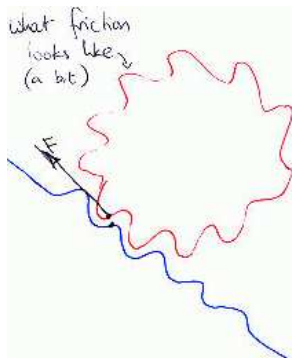
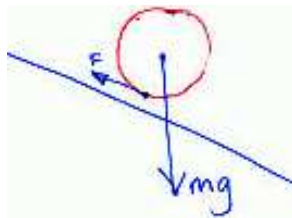
Miles per gallon =  $L / L^3 = L^{-2}$  ; it is an inverse-area.

The are corresponds to the cross-sectional area of the tube of petrol that the car would need to suck up if it is being refuelled continuously on its way.

e.g. 30 mpg  $\longleftrightarrow$  1/A, 1 gallon = 5 litre.

Area  $A = 0.1 \text{ mm}^2$ .

**D.13 Rolling.** Energy is conserved, because there is no relative motion at the contact point.



Fictional force does not do any work because there is no relative motion. Work done = force  $\times$  distance moved by point of application of force = 0.

#### D.14 Anharmonic potentials that are isochronous.

It is possible to integrate the equation of motion:

$$m\ddot{x} = -\frac{dV}{dx}$$

to etc then the (quarter) period ( $\tau$ ):

$$\tau \propto \int_0^A \frac{dx}{\sqrt{V(A) - V(x)}}$$

and we just want this to be independent of A; Can do a partial differentiation w.r. t. A to work this out?

Anyway, Answer: sheared/cropped/rotated version of the harmonic potential also works.

