

Precession of the Earth's polar axis

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Consider a point mass dm located at \mathbf{r} . A mass M at \mathbf{R} will exert a gravitational force

$$d\mathbf{F} = \frac{GM}{|\mathbf{R} - \mathbf{r}|^3}(\mathbf{R} - \mathbf{r})dm$$

on it, and so the torque about the origin of the coordinate system is

$$d\mathbf{C} = \mathbf{r} \times d\mathbf{F} = \frac{GM}{|\mathbf{R} - \mathbf{r}|^3} \mathbf{r} \times \mathbf{R} dm$$

If the magnitude of \mathbf{R} is much greater than that of \mathbf{r} , we can make a binomial expansion:

$$d\mathbf{C} \approx \frac{GM}{R^3} \left(1 + 3 \frac{\mathbf{r} \cdot \mathbf{R}}{R^2} \right) \mathbf{r} \times \mathbf{R} dm$$

Now choose a Cartesian coordinate system with the axes aligned with the principal axes of the Earth's moment of inertia ellipsoid, and integrate over the whole Earth. Thus

$$\begin{aligned} C_x &= \frac{GM}{R^3} \int \left(1 + 3 \frac{r_x R_x + r_y R_y + r_z R_z}{R^2} \right) (r_y R_z - r_z R_y) dm \\ &= \frac{3GM R_y R_z}{R^5} \int (r_y^2 - r_z^2) dm \\ &= \frac{3GM R_y R_z}{R^5} (I_{yy} - I_{zz}) \end{aligned}$$

and cyclic permutations thereof.

Effect of the Sun

To approximate the annual average torque due to the Sun, we can replace it by a uniform ring of mass M and radius R , oriented at an angle ε ($\approx 23^\circ$) to the xy -plane. Consider the element of this ring at $\mathbf{R} = R(\cos \phi, \sin \phi \cos \varepsilon, \sin \phi \sin \varepsilon)$ with mass $M d\phi/2\pi$, and integrate the expression for \mathbf{C} around the ring. This gives

$$C_x = \frac{3GM \cos \varepsilon \sin \varepsilon}{2R^3} (I_{yy} - I_{zz})$$

C_y and C_z are both zero.

Effect of the Moon

To first order, we can just repeat the above calculation with M and R replaced by the appropriate values for the Moon. Since

$$\frac{M_{\text{moon}}}{M_{\text{sun}}} \frac{R_{\text{sun}}^3}{R_{\text{moon}}^3} = 3.69 \times 10^{-8} \times 389^3 = 2.17$$

the total effect of the Sun and Moon should be about 3.17 times as great as that due to the Sun alone. To be slightly more sophisticated, we can attempt to allow for the fact that the Moon's orbit is inclined at about 5° or 0.09 radians to the plane of the ecliptic, which means that the Moon's 'mass ring' is spread out in a band from about 18° to 28° with respect to the plane of the equator. It isn't too hard to show that this reduces the average torque due to the Moon by a factor of about $(1 - 0.09^2) \approx 0.992$. Thus, the combined effect of Sun and Moon is expected to be about 3.15 times the effect of the Sun alone. Since (see below) we aren't going to do better than a 1% error in our estimate of the precession period, this refinement is probably unjustified. Nevertheless, I'll keep it in.

Calculating the precession period

Since the Earth's angular momentum vector precesses about the axis normal to the plane of the ecliptic, the precession rate is given by

$$\Omega = \frac{C}{I\omega \sin \varepsilon}$$

where ω is the Earth's rotational angular velocity and I its (polar) moment of inertia. The precession period is thus

$$T = \frac{1}{3.15} \frac{4\pi\omega R^3}{3GM \cos \varepsilon} \frac{I}{\Delta I}$$

Here, M and R are the values for the Sun; the factor of 3.15 deals with the Moon. ΔI is the difference between the polar and equatorial moments of inertia.

Since

$$\frac{GM}{R^3} = \omega_y^2$$

where ω_y is the Earth's angular velocity in its orbit around the Sun, we can write the precession period as

$$T = \frac{1}{3.15} \frac{2y}{3 \cos \varepsilon} \frac{I}{\Delta I} \quad \text{years}$$

where y is the number of days in a year. Taking $\varepsilon = 23.4^\circ$, $y = 365$ and $I/\Delta I = 300$ gives about 25,300 years for the precession period. This is within a couple of percent of the true value.

Estimating $\Delta I/I$ from the Earth's eccentricity

Perhaps I have cheated a bit by assuming the value of $I/\Delta I$. Can we estimate it? We assume the Earth is a uniform ellipsoid with equatorial radius a and polar radius b . The moments of inertia are

$$I_p = \frac{2}{5} Ma^2$$

and

$$I_e = \frac{1}{5} M(a^2 + b^2)$$

(is this obvious? It's quite a well known result) which gives

$$\frac{\Delta I}{I} = \frac{a^2 - b^2}{2a^2} = \frac{e^2}{2}$$

where e is the eccentricity, defined by

$$b^2 = a^2(1 - e^2)$$

This is correct to within about 2%.

Modelling the Earth's eccentricity

But perhaps I have still cheated by assuming that we know the Earth's eccentricity. Can we estimate this too? Unfortunately this is much harder, and probably not accessible at the level of IB Advanced Physics. Nevertheless, let's see what we can do.

'It can be shown that' the Earth's gravitational potential can be expanded in spherical harmonics as

$$V = -\frac{GM}{r} \left(1 - \sum_{n=2} J_n \left(\frac{a}{r} \right)^n P_n(\sin \phi) \right)$$

where ϕ is the latitude. M is now the Earth's mass, and a is still its equatorial radius. Retaining terms up to $n=2$, we have for a point on the equator

$$V_e = -\frac{GM}{a} \left(1 + \frac{J_2}{2} \right)$$

and for the North Pole

$$V_p = -\frac{GM}{b} \left(1 - \frac{J_2}{2} \left(\frac{a}{b} \right)^2 \right)$$

where b is the polar radius. Now

$$J_2 = \frac{I_p - I_e}{Ma^2}$$

(again, not going to prove that!) so approximating the mass distribution as a uniform ellipsoid, as before, we obtain

$$J_2 = \frac{e^2}{5}$$

(in fact this is a serious overestimate – by about 24% – because the mass distribution in the Earth is not uniform).

Setting

$$V_e - \frac{\omega^2 a^2}{2} = V_p$$

and making the substitutions

$$\frac{GM}{a^2} = g$$

$$\frac{b^2}{a^2} = 1 - e^2$$

we obtain the following friendly expression:

$$1 - \frac{e^2}{5(1-e^2)} - 1 - \frac{e^2}{10} = \frac{\omega^2 a}{2g}$$

The first term in the binomial expansion of the left-hand side is, however, just $e^2/5$, so we finally obtain our estimate of e^2 as

$$e^2 = \frac{5\omega^2 a}{2g}$$

Substituting the values gives $e^2 = 8.64 \times 10^{-3}$, which is an overestimate by about 30%. I don't think we can do any better than this unless we start putting in some geophysics.

Summary

We can estimate the precession rate to within a few percent of the right answer if we are allowed to assume the value of $\Delta I/I$ or (perhaps more reasonably) the Earth's eccentricity. However, if we want to estimate the value of $\Delta I/I$ to this sort of accuracy we need to know something about the internal structure of the Earth. I think I run out of ingenuity at this point!