

Warning: these solutions have not been proofread yet.

T.1 Pulleys

(a)

The starting point is to identify the constraint imposed by the geometry: $z_2 = -2z_1$. [Here, we've chosen the origin of z_1 and z_2 for our convenience.]

The energies are $T = \frac{m}{2}[\dot{z}_1^2 + \dot{z}_2^2]$ and $V = mg(z_1 + z_2) = -mgz_1$. We eliminate z_2 and use z_1 to describe our one degree of freedom. The energy is

$$E = T + V = \frac{m}{2}[5\dot{z}_1^2] - mgz_1 \quad (1)$$

By the energy method ($\frac{dE}{dt} = 0$),

$$5m\ddot{z}_1\dot{z}_1 = mg\dot{z}_1 \quad (2)$$

$$\Rightarrow \ddot{z}_1 = +\frac{g}{5}, \quad \ddot{z}_2 = -\frac{2g}{5}. \quad (3)$$

$$(b) \ E = \frac{1}{2}(M + m)\dot{z}^2 + \frac{1}{2}\frac{I}{a^2}\dot{z}^2 + (m - M)gz$$

$$\ddot{z} = \frac{(M - m)g}{\left(M + m + \frac{I}{a^2}\right)}$$

Solution to T2: Spring 1. The total energy is

$$E = \frac{1}{2}m\dot{z}^2 + mgz + \frac{1}{2}kz^2, \quad (4)$$

where z is the compression (if positive) or extension (if negative) of the spring, relative to its unstretched length. By the energy method,

$$m\ddot{z}\dot{z} + mg\dot{z} + kz\dot{z} = 0 \quad (5)$$

so the equation of motion is

$$\ddot{z} = -g - \frac{k}{m}z. \quad (6)$$

The vertical acceleration is zero when the right hand side is zero, *i.e.*, when $z = -mg/k$. So the mass can sit in equilibrium if it drops to a height such that the force from the spring (magnitude kz) is equal and opposite to the weight (magnitude mg). The solution of the equation of motion is sinusoidal oscillation about this equilibrium point:

$$z(t) = -mg/k + A\sin(\omega t + \phi), \quad (7)$$

where $\omega^2 = k/m$, and A and ϕ are determined by the initial conditions. [That this is solution is easy to see if we introduce the displacement from equilibrium, $x = z - (-mg/k)$, and find the equation of motion for x , which is $\ddot{x} = -k/mx$.] Notice the unstretched length l has appeared nowhere.

On the moon, the value of g is smaller. This changes the equilibrium point (mg/k is smaller, so the mass does not hang so low). But it has no effect on the frequency $\omega = \sqrt{k/m}$. One way of thinking about this is that changing g changes the linear term in the potential energy

$$V(z) = mgz + \frac{1}{2}kz^2, \quad (8)$$

but it has no effect on the quadratic term, and it is always quadratic terms in potentials that determine oscillation frequencies, since ('equation zero')

$$\omega^2 = \frac{\partial^2 V}{\partial z^2} \bigg/ m, \quad (9)$$

and when you differentiate twice, what you obtain is the coefficient of the quadratic term.

variable	dimension
m	M
g	LT^{-2}
k	MT^{-2}
l	L
ω	T^{-1}
5	3

By dimensional analysis, how far we can get depends on whether we include the unstretched length, which could, in principle, have some relationship to the period of small oscillations – indeed, if we use the spring and mass as a simple pendulum, then this length *will* appear in the expression for the period. We want to find how ω depends on the other variables; we need to find $5 - 3 =$ two dimensionless groups. One group is $(\omega^2 m/k)$. Another is (mg/kl) , which is the ratio of the weight of the mass to the force exerted by the spring when we double its length. From these two groups we can deduce that the dependence of ω must have the form

$$\omega = \left(\frac{k}{m}\right)^{1/2} F(mg/kl), \quad (10)$$

where F is a dimensionless function. This answer would leave open the possibility that the frequency *does* depend on the strength of gravity. However, if we further assume that there is no dependence on the unstretched length l (and you could argue for that by a thought experiment in which you replace the spring by another with identical k and different l), then dimensional analysis tells us that

$$\omega = \kappa \left(\frac{k}{m}\right)^{1/2} \quad (11)$$

where κ is a dimensionless constant that is *independent of the strength of gravity*.

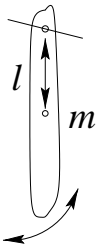
Pretty neat, hey? Purely on dimensional grounds, you can tell that the vertical oscillations have the same frequency on the moon and on the earth.

T.3 Compound pendulum

(a)

The moment of inertia about the axis is $I = I_0 + ml^2 = m(k^2 + l^2)$, and the total energy is $E = T + V = \frac{1}{2}I\dot{\theta}^2 + mgl(1 - \cos(\theta))$, so, by the energy method:

$$m(l^2 + k^2)\ddot{\theta} = -mgl \sin \theta.$$



For small θ , we approximate $\sin \theta \simeq \theta$ and get

$$\ddot{\theta} = -\frac{gl}{(l^2 + k^2)}\theta.$$

The solution of this equation is simple harmonic motion with frequency

$$\omega = \sqrt{\frac{gl}{k^2 + l^2}}$$

The period is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$

Interpretation: At small l , ($k^2 \gg l^2$) rotational inertia dominates, and the restoring couple (which scales as gl) becomes small because the centre of mass rises little when the pendulum is displaced; so for small l the period becomes large. At large l , the pendulum becomes like a simple pendulum, with period increasing with l . Thus for both large and small l , the period increases, and there is a minimum period at intermediate l . In order to sketch a graph of the period, we differentiate $k^2 + l^2/gl$ with respect to l and find that it is zero at $l = k$.

We replicate the graph of $T(l)$ for negative l using $T(l) = T(|l|)$ [the meaning of negative l is that we are suspending the pendulum from a point on ‘the other side of the centre of mass’]. The figure shows a sketch of T^2 versus l/k .

(b) Notice that any particular period can be obtained by suspending the pendulum from four different points, two close to the centre of mass, and two further away. Let’s define the two distances $|l|$ of these pairs of points to be l_A and l_B ; then we have

$$T = 2\pi\sqrt{\frac{k^2 + l_A^2}{gl_A}} = 2\pi\sqrt{\frac{k^2 + l_B^2}{gl_B}} \quad (12)$$

So can solve for k :

$$(k^2 + l_A^2)l_B = (k^2 + l_B^2)l_A$$

$$k^2(l_A - l_B) = l_A^2l_B - l_B^2l_A$$

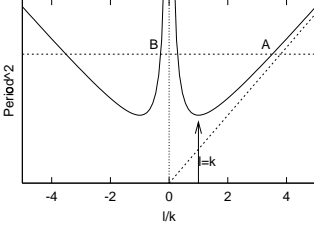
Dividing by $(l_A - l_B)$, which is legal because l_A and l_B are not equal, we obtain

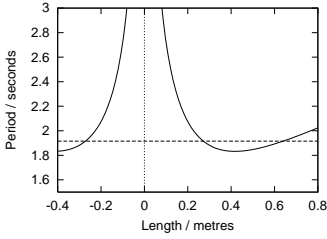
$$k^2 = l_A l_B$$

which when plugged into the period (12) gives

$$T = 2\pi\sqrt{\frac{l_A + l_B}{g}} \quad (13)$$

This formula is the basis of an accurate measurement of g : pick two pivot points A and B on opposite sides of the centre of mass, and at unequal distances such that the periods are equal. Then, to evaluate g , we only need to measure the separation of the two pivot points $|l_A| + |l_B|$ – we don’t need to know where the centre of mass is, and, we don’t need to know l_A and l_B individually. To make the measurement accurate, the pivots can be made from sharp knives, and to ensure the effects of air motion are identical in the two orientations, the pendulum can be symmetrical in shape.





variable	dimension
f	MLT^{-2}
m	M
g	LT^{-2}
k^*	MLT^{-2}
l	L
5	3

In practice, the two periods T_A and T_B will not be exactly equal, but they can both be measured accurately, and g can still be measured to one part in 10^4 with the help of estimates of the distances l_A and l_B .

The figure shows the period versus l for a Kater pendulum, about one metre long, at the Cavendish laboratory. The horizontal line shows the value of the period that can be obtained by putting the two knife edges as far apart as possible. See the course website for further experimental notes.

T.5 Safety rope

If we put two springs of constant k end to end, the resulting spring has spring constant equal to $k/2$, since for a given force its extension is twice as great. Thus a length l of stretchable rope has spring constant *inversely proportional* to l . We thus write $k = k^*/l$, where k^* is a property of the rope.

Using dimensional analysis, there are two dimensionless groups to be found: we can choose (f/mg) and (mg/k) . Thus we obtain

$$f \propto mg G\left(\frac{mg}{k^*}\right),$$

where G is a dimensionless function. Notice that this implies that f has *no dependence on l* . That's a pretty strong result!

The motion can be solved using energy conservation. The rope stretches until

$$\frac{k^* x^2}{2l} = m(l+x)g$$

where x is the extension; $(l+x)$ is the total length of the extended rope. This is a quadratic equation for x , in general. In the case of a stiff rope, the extension x will be small compared to the length l , so we can replace $(l+x)$ by l . We then find that the maximum force is $F = kx = \sqrt{2k^*mg}$. If the rope is very stretchable, then we expect x to be much greater than l , so we can replace $(l+x)$ by x . This gives $F = 2mg$.

T.6: Oscillation

The equation of motion (found by the energy method, for example) is

$$m\ddot{x} = -\frac{\partial V}{\partial x} = -\frac{A}{x^2} + \frac{12B}{x^{13}}. \quad (14)$$

The acceleration is zero at the $x = x_0$ such that

$$\frac{A}{x^2} = \frac{12B}{x^{13}}, \quad (15)$$

i.e.,

$$x_0^{11} = \frac{12B}{A}. \quad (16)$$

We now Taylor-expand V about this equilibrium point:

$$V \simeq V(x_0) + \frac{1}{2} \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} (x-x_0)^2 \dots \quad (17)$$

so the equation of motion for small deviations $x - x_0$ from equilibrium is

$$m\ddot{x} = - \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} (x - x_0) \quad (18)$$

which implies, if the second derivative is positive, simple harmonic motion with frequency

$$\omega^2 = \left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} / m. \quad (19)$$

Now, we can evaluate the second derivative by brute force:

$$\left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} = +2 \frac{A}{x_0^3} - \frac{13 \times 12B}{x_0^{14}}, \quad (20)$$

or we can use hygienic differentiationTM:

$$\left. \frac{\partial^2 V}{\partial x^2} \right|_{x=x_0} = \left. \frac{\partial}{\partial x} \left(\frac{1}{x^{13}} \right) (Ax^{11} - 12B) \right|_{x=x_0} \quad (21)$$

$$= \left. \left(\frac{\partial}{\partial x} \frac{1}{x^{13}} \right) (Ax^{11} - 12B) \right|_{x=x_0} + \frac{1}{x^{13}} \left. \left(\frac{\partial}{\partial x} Ax^{11} \right) \right|_{x=x_0} \quad (22)$$

$$= 0 + \frac{1}{x_0^{13}} (11Ax_0^{10}) \quad (23)$$

$$= \frac{11A}{x_0^3} \quad (24)$$

The oscillation frequency is thus given by

$$\omega^2 = \frac{1}{m} \frac{11A}{x_0^3} = \frac{11A}{m} \left(\frac{A}{12B} \right)^{3/11} \quad (25)$$

At this stage, it would be good to check dimensions. A has dimensions of energy times length, *i.e.*, ML^3T^{-2} . A/B has dimensions L^{-11} , so the right hand side has dimensions $L^3T^{-2}L^{-3} = T^{-2}$. Incidentally, the dependence of ω on A and B could have been deduced by dimensional analysis, except for the dimensionless constant.

Solution to T4: Wonky pendulum. Let's measure coordinates x and y from the centre of the cylinder.

$$y(\theta) = a \sin \theta - (l + a\theta) \cos \theta \quad (26)$$

$$x(\theta) = a \cos \theta + (l + a\theta) \sin \theta \quad (27)$$

From these we can evaluate the kinetic energy T and potential energy V . T can be found by computing $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\dot{\theta}^2[(dx/d\theta)^2 + (dy/d\theta)^2]$, leading to

$$T = \frac{1}{2}m(l + a\theta)^2\dot{\theta}^2, \quad (28)$$

which we can recognise as $\frac{1}{2}I\dot{\theta}^2$, where $I = m(l + a\theta)^2$ is the instantaneous moment of inertia about the point of suspension, and $\dot{\theta}$ is the instantaneous angular velocity about that point. With some confidence, therefore, we could have gone directly to the expression (28) without slaving through the ugly preceding step.

The potential energy is

$$V(\theta) = mgy(\theta) = mg[a \sin \theta - (l + a\theta) \cos \theta]. \quad (29)$$

When we use the energy method, we will need the derivative

$$\frac{\partial V(\theta)}{\partial \theta} = mg[a \cos \theta + (l + a\theta) \sin \theta - a \cos \theta] = mg[(l + a\theta) \sin \theta] \quad (30)$$

[Notice how similar this is to the derivative of the potential of an ordinary simple pendulum.] We now use the energy method

Tip: **don't** expand $(l + a\theta)^2$! Differentiate it as a whole.

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt} \quad (31)$$

$$= \overbrace{m(l + a\theta)\dot{\theta}^2 a\dot{\theta} + m(l + a\theta)^2 \dot{\theta}\ddot{\theta}} + mg[(l + a\theta) \sin \theta] \dot{\theta} \quad (32)$$

Dividing by $\dot{\theta}$ and rearranging, we find the equation of motion:

$$\ddot{\theta} = -\frac{g}{(l + a\theta)} \sin \theta - \frac{a}{(l + a\theta)} \dot{\theta}^2. \quad (33)$$

This equation demands interpretation. The first term on the right hand side is familiar: it is the angular acceleration of a simple pendulum with length $(l + a\theta)$. What about the second term? Let's take the **special case** where $g = 0$, so that this is the only term. Imagine an ice hockey puck on a piece of string, wrapping round a vertical pole as it slides on horizontal ice, for example.

As the puck goes round, does its angular momentum about the centre remain constant? What about its energy? Think about it, or try the experiment, and you'll agree that the magnitude of the angular velocity $\dot{\theta}$ *increases* as the puck winds up (*i.e.*, when θ is negative), and that it decreases as it unwinds. These two effects are captured by the term proportional to $-\dot{\theta}^2$.

Think about it. Once you are satisfied with the meaning of this second term, we can turn to the last part: *Find the period of small oscillations.*

For small θ , $a\theta$ is negligible compared with l , so all terms of the form $(l + a\theta)$ can be replaced by l ; also, $\sin \theta \simeq \theta$; and finally, what can we say about the term

$$-\frac{a}{l}\dot{\theta}^2?$$

Remember, this term is related to the changing length of the pendulum, which is small for small oscillations.

The equation of motion $\ddot{\theta} = -\omega^2\theta - \gamma\dot{\theta}^2$ may be unfamiliar to us, but we can make progress by trying a **bold assumption** that the $\gamma\dot{\theta}^2$ term is negligible, then solving the simpler equation $\ddot{\theta} = -\omega^2\theta$, then coming back and double-checking our assumption.

If we neglect the term in $\dot{\theta}^2$ then the motion is simple harmonic,

$$\theta = \theta_{\max} \sin(\omega t + \phi); \quad \dot{\theta} = \omega \theta_{\max} \cos(\omega t + \phi) \quad (34)$$

so the nonlinear term $\frac{a}{l}\dot{\theta}(t)^2$ scales as $\frac{a}{l}\omega^2\theta_{\max}^2$, whereas the linear term $\omega^2\theta(t)$ scales as $\omega^2\theta_{\max}$, so, for small θ_{\max} , the nonlinear term is negligible compared with the linear term, nearly all the time.

Conclusion: The period is $T = 2\pi\sqrt{\frac{l}{g}}$.

Comments: When the puck winds round the pole (when $g = 0$), only the energy is conserved; angular momentum of the puck about the origin is not conserved because the force acting on the puck has a couple about the origin.

[‘Nearly all the time’, because, for a tiny fraction of each period, whenever θ is between roughly $\pm\theta_{\max}^2$, the linear restoring force will be sufficiently close to zero that the nonlinear term will be bigger than it; but if θ_{\max} is small then these intervals will last such a short time that the motion will scarcely differ from simple harmonic.]

*See the history section of the website for links.

Wonky pendula a bit like this one were developed by Christiaan Huygens*, who patented the pendulum clock. The motivation for including two cheeks in a pendulum clock, one on each side of the pendulum (in our problem there’s only one cheek), is that for an appropriate choice of the cheek, the variation of period with amplitude can be reduced. A good (hard) rider to this problem is to find the variation of period with amplitude of a two-cheek wonky pendulum, and find if there is a value of a/l such that the period has only an order θ_{\max}^4 dependence on amplitude, instead of the normal θ_{\max}^2 dependence of the simple pendulum.

A further rider is to find the shape of cheeks such that the pendulum has *exactly the same period for all amplitudes*. Huygens solved this problem: the beautiful answer is that the cheeks should be *cycloids*, and the path followed by the pendulum bob is also a cycloid!

As an encore, we can solve the wonky pendulum by Lagrangian methods. $L = T - V$. The conjugate momentum is:

$$\frac{\partial L}{\partial \dot{\theta}} = m(l + a\theta)^2 \dot{\theta}, \quad (35)$$

which is equal both to the angular momentum about the origin and the angular momentum about the instantaneous point of suspension. The generalized force is:

$$\frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} = m(l + a\theta)\dot{\theta}a\dot{\theta} - mg(l + a\theta)\sin\theta. \quad (36)$$

We now write the equation of motion, $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$:

$$2m(l + a\theta)\dot{\theta}^2 a + m(l + a\theta)^2 \ddot{\theta} = m(l + a\theta)\dot{\theta}^2 a - mg[(l + a\theta)\sin\theta] \quad (37)$$

Once we group together the terms in $\dot{\theta}^2$ that have turned up on both sides of this equation, we find that it agrees with the equation of motion (32).

The cycloid is the path followed by a piece of chewing gum on the tyre of a bicycle. Turn it upside down to get the path for the ‘isochronous’ pendulum’s bob.

T.9: Snooker

Qualitative description: If hit along the centre, the ball would immediately post-impulse have linear momentum and no angular momentum about its centre of mass; it would therefore be *slipping*. The friction decelerates the linear motion, and exerts a couple about the centre of mass, causing the ball to rotate. As it rotates faster and moves more slowly, a point will come when the linear velocity and angular velocity are compatible, so the ball starts to roll without slipping.

Now, assuming the standard model of friction, the frictional force that opposes the sliding is a constant (F), independent of the relative velocity. (The friction force is proportional to the perpendicular force, but that is not varying in this problem.)

$$m\dot{v} = -F \longrightarrow v = v_0 - \frac{F}{m}t$$

$$I\dot{\omega} = Fr \longrightarrow r\omega = 0 + \frac{Fr^2}{I}t = \frac{5F}{2m}t$$

After $t_0 = \frac{2mv_0}{7F}$, $v = \omega r$, at which point it starts rolling.

To make it roll right away, the momentum impulse Δp should be such that it sets up compatible linear motion and rotation. Therefore

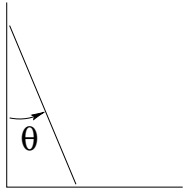
$$h\Delta p = I\omega = \frac{2}{5}ma^2\omega$$

$$\Delta p = mv$$

Using $v = a\omega$ we obtain $h = \frac{2a}{5}$ i.e. hit $\frac{7a}{5}$ above table, or 70% the full height of the ball.

Lagrangian and Hamiltonian dynamics

T.10. Ladder



The ladder has length $2l$. The potential energy is $V = -mgl(1 - \cos \theta)$. The kinetic energy can be written as the sum of the rotational kinetic energy plus the translational energy of the centre of mass: $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. Now, as you can confirm with a sketch, the centre of mass moves along a circular path centred on the origin, and $(\dot{x}^2 + \dot{y}^2) = l^2\dot{\theta}^2$. So the Lagrangian is

$$L = T - V = \frac{1}{2}(I + ml^2)\dot{\theta}^2 + mgl(1 - \cos \theta). \quad (38)$$

Notice that this Lagrangian is *identical* to that of the compound pendulum, except that the potential energy term has flipped sign. Thus the falling ladder is equivalent to an upside-down compound pendulum.

The equation of motion is

$$(I + ml^2)\ddot{\theta} = +mgl \sin \theta. \quad (39)$$

Solution to T11: Pulley Galore. This system has two degrees of freedom. Let's answer the final question first. The most useful extreme case to think about is where

the right hand masses are replaced by $4m$ and ϵm . In this limit, both the $4ms$ plummet towards the ground at g , and the little guy goes up at $3g$; the tension in all the strings is negligible. Similarly, in the given case, the tension in the right hand string is less than the $2mg$ that would be needed to balance the $4mg$ weight on the left because the $3mg$ mass is quite close to a state of free fall, so it's not pulling its weight.

Let's use as our coordinates z_4 , the height of the $4m$, and z_2 , the distance through which the right-hand pulley rotates. Thus the height of the $3m$ is defined to be $z_2 - z_4$, and of that of the m is $-z_2 - z_4$. The Lagrangian is

$$L = T - V \quad (40)$$

$$= \frac{1}{2}4m\dot{z}_4^2 + \frac{1}{2}3m(\dot{z}_2 - \dot{z}_4)^2 + \frac{1}{2}m(\dot{z}_2 + \dot{z}_4)^2 - 4mgz_4 - 3mg(z_2 - z_4) - mg(-z_2 - z_4) \quad (41)$$

$$= 4m\dot{z}_4^2 + 2m\dot{z}_2^2 - 2m\dot{z}_4\dot{z}_2 - 2mgz_2 \quad (42)$$

The conjugate momenta are

$$p_4 = 8m\dot{z}_4 - 2m\dot{z}_2 \quad (43)$$

$$p_2 = 4m\dot{z}_2 - 2m\dot{z}_4 \quad (44)$$

The Euler-Lagrange equations are

$$\frac{d}{dt}[8m\dot{z}_4 - 2m\dot{z}_2] = 0 \quad (45)$$

$$\frac{d}{dt}[4m\dot{z}_2 - 2m\dot{z}_4] = -2mg \quad (46)$$

Rearranging, we can solve for the two accelerations.

$$8\ddot{z}_4 - 2\ddot{z}_2 = 0 \quad (47)$$

$$-\ddot{z}_4 + 2\ddot{z}_2 = -g \quad (48)$$

$$7\ddot{z}_4 = -g \Rightarrow \ddot{z}_4 = -g/7 \quad (49)$$

$$\ddot{z}_2 = -g/2 - g/14 = -4g/7. \quad (50)$$

So the big guy falls with acceleration $g/7$, and the $3m$ falls at $3g/7$, and the smallest mass accelerates upwards at $5g/7$.

Solution to T12: Vertical state space. $L = \frac{1}{2}m\dot{z}^2 - mgz$. $p = m\dot{z}$. $H = p\dot{z} - L = \frac{1}{2}\frac{p^2}{m} + mgz$. Hamilton's equations are:

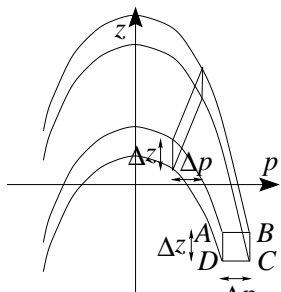
$$\frac{d}{dt}z = \frac{p}{m}; \quad \frac{d}{dt}p = -mg. \quad (51)$$

The solution for $z(t)$ and $p(t)$ is

$$p(t) = p(0) - mg \quad (52)$$

(momentum is a linear decreasing function of time);

$$z(t) = z(0) + ut - \frac{1}{2}gt^2, \quad (53)$$



Solution to T13:

I assume that the spring is unstretched when z_1 and z_2 are both zero. The extension of the spring is defined to be $-(z_1 + z_2)$.

where $u = p(0)/m$. Since z is a parabolic function of time and p is linear with time, z is also a parabolic function of p .

From the solution for p , (52), we can see that two initial conditions that differ from each other by Δp will lead to later states that still differ by exactly Δp . From (53), we can see that initial differences in z alone will lead to equal differences in z later. And from the ut term in (53), we can see that initial differences in p will cause growing vertical differences. So the rectangle ABCD evolves into a parallelogram. But the area of the parallelogram is still $\Delta p \Delta z$.

Spring pulley. Unfortunately in my original solution I measured z_1 upwards and z_2 downwards. Anyone using the sensible ‘both upwards’ convention may have found my hint unhelpful since it has the wrong sign for z_2 – sorry!

In the special case $g = 0$, the masses perform simple harmonic motion about points that move at constant velocity. If the spring is replaced by a string and g is non-zero then the two masses accelerate at rate $g(m_1 - m_2)/(m_1 + m_2)$ with m_1 accelerating down.

We write down the Lagrangian, using the ‘both upwards’ convention:

$$L = T - V = \frac{1}{2}m_1\dot{z}_1^2 + \frac{1}{2}m_2\dot{z}_2^2 - \frac{1}{2}k(z_1 + z_2)^2 - g(m_1z_1 + m_2z_2). \quad (54)$$

Since this is not a driven system, the Hamiltonian comes out to be $H = T + V$. We rewrite it in terms of the momenta p_i .

$$H = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + \frac{1}{2}k(z_1 + z_2)^2 + g(m_1z_1 + m_2z_2). \quad (55)$$

Hamilton’s equations are

$$\dot{x}_1 = \frac{p_1}{m} \quad (56)$$

$$\dot{x}_2 = \frac{p_2}{m} \quad (57)$$

$$\dot{p}_1 = -k(z_1 + z_2) - m_1g \quad (58)$$

$$\dot{p}_2 = -k(z_1 + z_2) - m_2g. \quad (59)$$

$$(60)$$

These last two equations can be recognized as describing the forces acting on the two masses. However, it’s not obvious what the solution of these equations is, nor what the conserved quantities are.

If we introduce

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ m_1/M & -m_2/M \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad (61)$$

where $M = m_1 + m_2$, then we can rewrite the kinetic energy and the potential energy.

First the kinetic energy. It is useful to have the inverse relationship (found by standard matrix inversion)

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} m_2/M & 1 \\ m_1/M & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

so the velocities are given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} m_2/M & 1 \\ m_1/M & -1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}$$

and

$$T = \frac{1}{2} \begin{bmatrix} \dot{z}_1 & \dot{z}_2 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \quad (62)$$

$$= \frac{1}{2} \begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix} \begin{bmatrix} m_2/M & m_1/M \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} m_2/M & 1 \\ m_1/M & -1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \quad (63)$$

$$= \frac{1}{2} \begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix} \begin{bmatrix} m_2/M & m_1/M \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_1 m_2/M & m_1 \\ m_1 m_2/M & -m_2 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \quad (64)$$

$$= \frac{1}{2} \begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix} \begin{bmatrix} m_1 m_2/M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \quad (65)$$

$$= \frac{1}{2} \mu \dot{u}_1^2 + \frac{1}{2} M \dot{u}_2^2 \quad (66)$$

$$(67)$$

where $\mu \equiv m_1 m_2 / M$ is the standard reduced mass.

Now the potential energy. The spring term $\frac{1}{2} k (z_1 + z_2)^2$ is simply $\frac{1}{2} k u_1^2$. The gravitational term $g(m_1 z_1 + m_2 z_2)$ requires a little more attention.

$$g(m_1 z_1 + m_2 z_2) = g \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (68)$$

$$= g \begin{bmatrix} m_1 & m_2 \end{bmatrix} \begin{bmatrix} m_2/M & 1 \\ m_1/M & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (69)$$

$$= g \begin{bmatrix} 2m_1 m_2 / M & (m_1 - m_2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (70)$$

$$= 2\mu g u_1 + (m_1 - m_2) g u_2 \quad (71)$$

$$(72)$$

Of these two terms, the second corresponds to simple increase in gravitational potential as the variable u_2 , which describes the motion of the effective centre of mass, increases. The factor $(m_1 - m_2)$ makes sense because if the masses are equal then the gravitational potential energy is independent of this translation.

The first term is also intuitive: imagine the two masses hanging over the pulley, and let's say the masses are equal. Will the spring be extended? Of course, because they are both hanging down from the spring. Thus the gravitational potential couples to u_1 as well as u_2 . The effect of this linear term in u_1 is just like the effect of gravity on an ordinary mass-spring system: it shifts the equilibrium point to a non-zero value of the coordinate.

We introduce the new momenta $p'_1 = \mu \dot{u}_1$ and $p'_2 = M \dot{u}_2$.

The new Hamiltonian is

$$H = T + V = \frac{1}{2\mu} p_1'^2 + \frac{1}{2M} p_2'^2 + \frac{1}{2} k u_1^2 + 2\mu g u_1 + (m_1 - m_2) g u_2 \quad (73)$$

from which the equations of motion are

$$\dot{p}'_1 = -ku_1 - 2\mu g \quad (74)$$

$$\dot{p}'_2 = -(m_1 - m_2)g \quad (75)$$

$$(76)$$

or, in terms of the coordinate accelerations,

$$\ddot{u}_1 = -\frac{k}{\mu}u_1 - 2g \quad (77)$$

$$\ddot{u}_2 = -\frac{m_1 - m_2}{M}g \quad (78)$$

$$(79)$$

Sanity check this: if masses are equal, then the equilibrium extension should be mg/k , and $\mu = m/2$. Good.

Thus u_1 performs simple harmonic motion about the equilibrium value

$$u_1^* = -\frac{2g}{k/\mu}, \quad (80)$$

and u_2 accelerates at a rate proportional to the mass difference.

Solution to T14: Merry.

There is one degree of freedom, the angle θ . The child controls $r(t)$. If you try playing on a roundabout you will know that moving into the centre requires work, so the system does not have conserved energy. [Of course, the total energy of the *universe* is constant; when I say that the system's energy is not constant, I am referring to the kinetic energy of the child and the roundabout. If the child heaves herself towards the centre of the roundabout, she's doing work, and the chemical energy in her muscles is decreasing, exactly as the kinetic energy of the child-and-roundabout increases.]

The potential V is zero. So the Lagrangian is

$$L = T = \frac{1}{2}m\dot{r}(t)^2 + \frac{1}{2}(mr(t)^2 + I)\dot{\theta}^2, \quad (81)$$

where I is the moment of inertia of the roundabout alone. Note carefully that r is time-dependent, so the Lagrangian is time-dependent. We could emphasise what is going on by writing the arguments of L : $L(\theta, \dot{\theta}, t)$. The first term, $\frac{1}{2}m\dot{r}(t)^2$ actually has no effect on the dynamics since it has no dependence on p_θ or θ , so it could be omitted.

The angular momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = (mr(t)^2 + I)\dot{\theta}. \quad (82)$$

The equation of motion is

$$\frac{d}{dt} [(mr(t)^2 + I)\dot{\theta}] = \frac{\partial L}{\partial \theta} = 0, \quad (83)$$

from which we can conclude the obvious fact that the angular momentum is a constant.

$$(mr(t)^2 + I)\dot{\theta} = J \quad (84)$$

The Hamiltonian is

$$H = p_\theta \dot{\theta} - L = \frac{1}{2}(mr(t)^2 + I)\dot{\theta}^2 - \frac{1}{2}m\dot{r}(t)^2. \quad (85)$$

The term $-\frac{1}{2}m\dot{r}(t)^2$ has no effect on the dynamics since it has no dependence on p_θ or θ , so, for practical purposes the Hamiltonian, the kinetic energy, the total energy, and the Lagrangian are all the same. We rewrite the Hamiltonian in terms of p :

$$H = \frac{1}{2} \frac{p_\theta^2}{mr(t)^2 + I} - \frac{1}{2}m\dot{r}(t)^2. \quad (86)$$

Hamilton's equations are:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr(t)^2 + I} \quad (87)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad (88)$$

$$(89)$$

As already stated, if r is time-varying, the Lagrangian is time-varying, so dH/dt is not zero. Neither the total kinetic energy nor the Hamiltonian is conserved.

Matrices

T.15 Displaced springs

Potential $V = \frac{1}{2}x_i K_{ij} x_j$, where

$$K_{ij} = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{pmatrix}$$

Assuming all the k_i 's are the same ($= k$),

$$F_1 = (kx_1 + k(x_1 - x_2)) = \frac{4}{3}kx_1$$

so

$$x_2 = \frac{2}{3}x_1 = \frac{F_1}{2k}$$

$$F_2 = (k(x_2 - x_1) + k(x_2 - x_3)) = kx_2$$

and

$$x_1 = \frac{x_2}{2} = \frac{F_2}{2k}$$

So indeed $x_j(F_i)$ has the same dependence as $x_i(F_j)$.

For the general case, the resisting force to external influence is $F_i = -\frac{\partial V}{\partial x_i} = -K_{ij}x_j$, so the external force is $w_i = -F_i = K_{ij}x_j$. If it is a stable dynamical system (i.e. it doesn't fly apart for the least force you apply onto any part of it), the K_{ij} should be invertable to give $x_i = (K^{-1})_{ij}w_j$. In fact, for coupled spring system, K_{ij} is symmetrical, and so is $(K^{-1})_{ij}$ (property of symmetry matrix). Hence $x_i = (K^{-1})_{ij}w_j$ is related to $x_j = (K^{-1})_{ji}w_i$ by the same ratio $(K^{-1})_{ij} = (K^{-1})_{ji}$.

T.17 3D spring

There are three degrees of freedom. By symmetry, the three modes must be one along the line of the springs (with frequency $2k/m$, independent of l and l_0) and two degenerate modes perpendicular to the springs. We find the frequency of the perpendicular modes by Taylor-expanding the potential. If the lateral displacement is z , then

$$V(z) = 2\frac{1}{2}ke^2 = k((z^2 + l^2)^{1/2} - l_0)^2; \quad (90)$$

the first derivative is

$$\frac{\partial V}{\partial z} = 2k((z^2 + l^2)^{1/2} - l_0)\frac{1}{2}(z^2 + l^2)^{-1/2}2z = 2k((z^2 + l^2)^{1/2} - l_0)(z^2 + l^2)^{-1/2}z. \quad (91)$$

The second derivative at $z = 0$, which is what we need to find the frequency of the mode, is

$$\left.\frac{\partial^2 V}{\partial z^2}\right|_{z=0} = 2k((z^2 + l^2)^{1/2} - l_0)(z^2 + l^2)^{-1/2}\Big|_{z=0} = 2k(l - l_0)/l. \quad (92)$$

[Note that we can be hygienic when differentiating: there are many z -dependent terms in the first derivative (91), but one of them (z) is zero at $z = 0$, so we only need to differentiate that one – the others don't matter.]

The expression we have derived is simply twice the tension $k(l - l_0)$ divided by the length l – a familiar result? So the frequency of the transverse modes is given by

$$\omega^2 = 2(k/m)(l - l_0)/l. \quad (93)$$

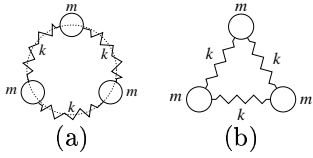
If the ‘stretched’ length l is smaller than the unstretched length l_0 , then ω^2 is negative. This means the fixed point is *unstable* to the two transverse displacements. If perturbed from equilibrium, the transverse displacement grows exponentially. To be precise, it grows initially exponentially, but once the displacement becomes large (*i.e.*, at all comparable to l), higher terms in the Taylor expansion become relevant. There will be a new stable equilibrium state, indeed, a whole circle of such states, in which the system is bent in a V shape and both springs have their unstretched lengths.

When the three springs are arranged symmetrically, there is by symmetry one transverse mode (in and out of the page), and its frequency, generalizing the two spring result, is $3(k/m)(l - l_0)/l$. As for the remaining two modes, it must be possible to find a pair that respect the three-fold symmetry of the system. The eigenvectors of the operator that rotates the in-plane displacement through 120 degrees are $(1, i)$ and $(1, -i)$ (see below for proof, if this is not familiar), and so these are normal modes of the three-spring system. They describe clockwise and anticlockwise circular motions. The two modes are degenerate, so any linear combination of them is a normal mode. Thus any displacement in the plane is a normal mode.

The potential, to quadratic order, has the form $\frac{1}{2}\mathbf{xKx}$; and this quadratic function must be invariant under rotation of \mathbf{x} through 120 degrees. The

only quadratic functions having this symmetry are ones in which \mathbf{K} is proportional to the identity matrix. The potential is thus $\frac{1}{2}k^*(x^2 + y^2)$, where x and y are the two in-plane displacements and k^* is the effective spring constant, which looks to me like $\sqrt{3}k$, but I should check it.

If the mass is given a kick in any direction, starting from the origin, it simply oscillates to and fro in that direction at frequency $\sqrt{k^*/m}$.



T19: Triangles

Three masses moving on a circle can be solved using the same method as the four masses in a circle – see the normal modes handout. (Alternatively, you can use guessing'n'checking.) The system is symmetric under clockwise permutation of the three displacements (*i.e.*, rotation through 120 degrees), so we can find the normal modes by finding the eigenvectors of that permutation operator. The $N = 3$ eigenvectors $\mathbf{f}^{(a)}$ are given by $f_n^{(a)} = e^{i2\pi an/N}$, for $a = 0, 1, 2$. The modes are

- $(1, 1, 1)$, which corresponds to steady rotation – this mode has zero frequency.
- $(1, e^{i2\pi/3}, e^{-i2\pi/3})$ and $(1, e^{-i2\pi/3}, e^{i2\pi/3})$. These modes correspond to complex travelling waves travelling clockwise and anticlockwise.

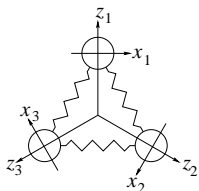
If we prefer all our modes to be real, we can take appropriate linear combinations of the two complex modes. Adding and subtracting, we obtain:

- $(2, -1, -1)$;
- $(0, 1, -1)$.

These modes are degenerate and both have frequency $\sqrt{3k/m}$.

(b) In the plane there are six degrees of freedom. Two of these correspond to translational degrees of freedom with no restoring force. A third is rotation, as before, a zero frequency mode. A new vibrational mode is a coherent contraction/dilation of all three masses. Two modes remain to be identified. They must be similar to the two complex modes of part (a). Are they identical to them? We know that it must be possible to find modes that respect the three-fold symmetry.

Let's try the *guessing* method, and see if the two modes of triangle (a) fit the bill. We know that the six modes must span the space and must be orthogonal. This will be sufficient to prove if the modes are correct. If we introduce coordinates $(x_1, z_1, x_2, z_2, x_3, z_3)$, the modes we have identified so far are: [here I define $c \equiv \cos 30 = \sqrt{3}/2$ and $s \equiv \sin 30 \text{ degrees} = 1/2$.



- translation in x_1 direction: $(1, 0, -s, c, -s, -c)$;
- translation in z_1 direction: $(0, 1, -c, -s, c, -s)$;
- rotation: $(1, 0, 1, 0, 1, 0)$.
- dilation: $(0, 1, 0, 1, 0, 1)$.

We now guess $(2, 0, -1, 0, -1, 0)$ and $(0, 0, 1, 0, -1, 0)$. We need to check their inner products with all the other modes. The nontrivial ones are the translations; does the centre of mass stay stationary in these modes? The inner product of candidate $(2, 0, -1, 0, -1, 0)$ with x_1 -translation $(1, 0, -s, c, -s, -c)$ is $+3$. This is not zero! So $(2, 0, -1, 0, -1, 0)$ is NOT a mode. Similarly, candidate $(0, 0, 1, 0, -1, 0)$ and z_1 -translation $(0, 1, -c, -s, c, -s)$ have inner product $-2c$.

We can refine our guesses by subtracting out these centre-of-mass motions. Our modified guesses are $(2 - 1, 0, -1 + s, 0 - c, -1 + s, 0 + c) = (1, 0, -s, -c, -s, +c)$ and $(0, 0 + 2c/3, 1 - 2c^2/3, 0 - 2sc/3, -1 + 2c^2/3, 0 - 2sc/3) = (0, 1/\sqrt{3}, 1/2, -1/(2\sqrt{3}), -1/2, -1/(2\sqrt{3}))$. Rescaling, $(0, 1, \sqrt{3}/2, -1/2, -\sqrt{3}/2, -1/2) = (0, 1, c, -s, -c, -s)$. These two vectors span the remaining subspace and are orthogonal to the first four modes, therefore they must be normal modes of the system.

We can also use these symmetry and orthogonality arguments to construct the modes graphically. Let's assume we want to create a mode similar to the mode $(0, 1, -1)$ of triangle (a), that is, a mode that is symmetric under reflection about the line through mass 1. We anticipate that the first mass must move vertically in this mode. Let it move through 1 unit. The question then is, where should the other two masses be placed? Our reasoning proceeds in two steps.

First, the mode must be orthogonal to the all-radial mode, and the projection onto the all-radial mode of the displacement of mass 1 is 1 unit. Therefore both displacements of the other two masses must lie on radial displacements of $-\frac{1}{2}$ unit. These two constraints are shown by dashed lines.

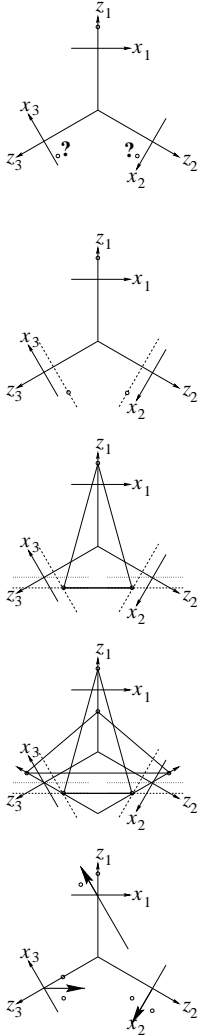
Second, the mode must be orthogonal to uniform centre-of-mass motion in the z_1 direction, so the displacements of the bottom two masses must both have projection $-\frac{1}{2}$ onto that direction. The lines on which the projection is zero are shown by dotted lines and the two new constraints are shown by dashed lines. Each of the masses can thus be pinned down to the intersections of its two dashed lines, as shown.

The next figure shows the two extremes of the mode that we have constructed.

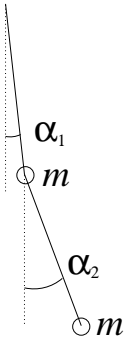
By adding together this mode, and its 120-degree-rotated twin, we can construct the other mode, the one that is analogous to the $(2, -1, -1)$ mode of triangle (a). The two twins that we add are represented by the small circles; the sums of the displacements are shown by the arrows. It is hard to imagine that one could guess that this particular displacement is a normal mode! Aren't symmetries neat?

T.20: Driven system

Method: Project the state (x_1, x_2) onto the eigenvectors $(1, 1)$ and $(1, -1)$, and work out the equation of motion for the projections u_1 and u_2 . The force acting on the first mass is equivalent to a force acting on each degree of freedom u_1 and u_2 .



T 21: Double pendulum



Let's first recap the dynamics of *the non-rotating* double pendulum. We assume the two masses are equal. We find the equation of motion by Lagrangian methods. Because we are interested in the motion *near the fixed point* $(\alpha_1, \alpha_2) = (0, 0)$, we will *approximate* the Lagrangian, making an approximation accurate for small angles.

For small angles, the masses' kinetic energy is associated almost entirely with horizontal motion; the two horizontal speeds are approximately $l\dot{\alpha}_1$ and $l\dot{\alpha}_1 + l\dot{\alpha}_2 = l(\dot{\alpha}_1 + \dot{\alpha}_2)$. So the kinetic energy is

$$T_{\text{not rotating}} \simeq \frac{1}{2}ml^2\dot{\alpha}_1^2 + \frac{1}{2}ml^2(\dot{\alpha}_1 + \dot{\alpha}_2)^2 = \frac{1}{2}ml^2 [2\dot{\alpha}_1^2 + 2\dot{\alpha}_1\dot{\alpha}_2 + \dot{\alpha}_2^2]. \quad (94)$$

The potential energy is

$$\begin{aligned} V &= mgl(1 - \cos \alpha_1) + mgl(1 - \cos \alpha_1 + 1 - \cos \alpha_2) \\ &= 2mgl(1 - \cos \alpha_1) + mgl(1 - \cos \alpha_2). \end{aligned} \quad (95)$$

For small angles, we can use $\cos \alpha \simeq 1 - \frac{1}{2}\alpha^2 + \dots$ to obtain

$$V \simeq 2mgl\frac{\alpha_1^2}{2} + mgl\frac{\alpha_2^2}{2}. \quad (96)$$

Notice that both these approximated energies can be written as quadratic forms:

$$T_{\text{not rotating}} = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix}. \quad (97)$$

$$V = \frac{1}{2}mgl \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \quad (98)$$

The effect of rotation at rate ω is to add extra terms to the kinetic energy. The radial distance of the first mass from the axis is $l\alpha_1$, so the rotational kinetic energy is $\frac{1}{2}ml^2\alpha_1^2\omega^2$; for the second mass, the extra energy is $\frac{1}{2}ml^2(\alpha_1 + \alpha_2)^2\omega^2$. So the total kinetic energy is

$$T = \frac{1}{2}ml^2 \begin{bmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} + \frac{1}{2}ml^2\omega^2 \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad (99)$$

and the Lagrangian of the rotating double pendulum is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}ml^2 \begin{bmatrix} \dot{\alpha}_1 & \dot{\alpha}_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} - \frac{1}{2}mgl \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} 2g - 2\omega^2l & -\omega^2l \\ -\omega^2l & g - \omega^2l \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \end{aligned} \quad (100)$$

We can think of these two quadratic forms as an effective kinetic energy T_{eff} and an effective potential V_{eff} , if we wish. We define \mathbf{M} and \mathbf{K} to be the two matrices in (101). We now solve for the generalized eigenvectors. In general, this would be a rather messy business, with solutions of quadratic equations running around. But in this case, the close relationship between the matrix proportional to ω^2 appearing in T , and the other matrix in T , means that it comes out rather nicely – the eigenvectors will be the same for all ω .

We want to find the eigenvalues, *i.e.*, the roots of

$$|\mathbf{K} - \lambda \mathbf{M}| = 0. \quad (102)$$

We divide through by ml^2 and define $\omega_0^2 = g/l$. [Not to be confused with ω^2 ; or λ , which will be the square of a normal mode frequency!]

$$\begin{vmatrix} 2\omega_0^2 - 2\omega^2 - 2\lambda & -\omega^2 - \lambda \\ -\omega^2 - \lambda & \omega_0^2 - \omega^2 - \lambda \end{vmatrix} = 0 \quad (103)$$

Since every ω^2 is accompanied by a λ , we define $\lambda' = (\omega^2 + \lambda)/\omega_0^2$, so we can save ink and solve

$$\begin{vmatrix} 2 - 2\lambda' & -\lambda' \\ -\lambda' & 1 - \lambda' \end{vmatrix} = 0, \quad (104)$$

finding

$$\lambda' = 2 \pm \sqrt{2} \quad (105)$$

Thus the frequencies of the two normal modes are given by

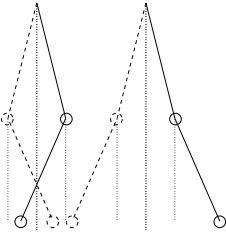
$$\lambda^{1/2} = (\lambda' \omega_0^2 - \omega^2)^{1/2} = \sqrt{(2 \pm \sqrt{2}) g/l - \omega^2}. \quad (106)$$

(a) For the special case of no rotation ($\omega = 0$), these frequencies are $1.8\sqrt{g/l}$ and $0.8\sqrt{g/l}$. The corresponding displacements are given by

$$\begin{bmatrix} 2 - 2\lambda' & -\lambda' \\ -\lambda' & 1 - \lambda' \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0, \quad (107)$$

from which we can find the ratio of e_1 to e_2 , giving

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \propto \begin{bmatrix} (-1 \mp \sqrt{2}) \\ (2 \pm \sqrt{2}) \end{bmatrix} = \begin{bmatrix} -2.4 \\ 3.4 \end{bmatrix} \text{ and } \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \quad (108)$$



Notice that these two eigenvectors are not orthogonal. The lower angle α_2 is bigger in magnitude in both modes. You might check that they do satisfy the generalized orthogonality rule. The mode with higher frequency is shown on the left, and the lower frequency to its right.

(b) For general rotation rate ω , the eigenvector equation (107) still applies, so *the eigenvectors are the same for all ω* . The only thing that changes is the frequency (106) of each normal mode. Since both frequencies decrease with ω , there will come a critical rotation rate at which the *lower* normal mode eigenvalue, $\lambda_- = (2 - \sqrt{2}) g/l - \omega^2$, will change sign. The equation of motion for the displacement of that normal mode coordinate will therefore change from

$$\ddot{x}_a = -\omega^2 x_a \quad (109)$$

to

$$\ddot{x}_a = |\lambda_-| x_a, \quad (110)$$

whose solutions are exponentially growing and decaying functions, rather than oscillatory functions. If ω^2 exceeds the critical value, $(2 - \sqrt{2}) g/l$, the fixed point changes from a stable to an unstable fixed point. For any

perturbation from the fixed point, the amplitude of the component of the lower frequency normal mode will grow exponentially.

If an uncle holds a niece in the air and spins her round, there is a critical spinning rate above which the niece tends to fly round with α_1 and α_2 both large and positive.

Rotating pendulum. There is one degree of freedom, α , and the Lagrangian is

$$L(\alpha, \dot{\alpha}) = \frac{1}{2}ml^2\dot{\alpha}^2 + \frac{1}{2}m(l\sin\alpha)^2\omega^2 - mgl(1 - \cos\alpha) \quad (111)$$

It may be convenient to group the three terms thus:

$$T_{\text{eff}} \equiv \frac{1}{2}ml^2\dot{\alpha}^2; \quad V_{\text{eff}} \equiv -\frac{1}{2}m(l\sin\alpha)^2\omega^2 + mgl(1 - \cos\alpha) \quad (112)$$

Expanding the effective potential to quadratic order about $\alpha = 0$,

$$V_{\text{eff}} \simeq -\frac{1}{2}ml^2\omega^2\alpha^2 + \frac{1}{2}mgl\alpha^2 = \frac{1}{2}ml^2(g/l - \omega^2)\alpha^2, \quad (113)$$

we are able to find the motion near to that point: it is simple harmonic motion with frequency Ω_{SHM} given by

$$\Omega_{\text{SHM}}^2 = g/l - \omega^2, \quad (114)$$

as long as Ω_{SHM}^2 is positive; if it's negative, then the fixed point is unstable, and the motion, for small deviations from $\alpha = 0$, is a sum of two exponentials with growth rates $\pm\sqrt{|g/l - \omega^2|}$.

Notice that the frequency of SHM about $\alpha = 0$ decreases as ω increases, approaching zero at $\omega^2 = g/l$.

When ω exceeds the critical value $\omega_{\text{crit}} = \sqrt{g/l}$, the point $\alpha = 0$ becomes unstable, and our quadratic approximation of the Lagrangian becomes inadequate to describe the situation. $\sin\alpha = \alpha - \alpha^3/6 + \dots$; so $(\sin\alpha)^2 = \alpha^2 - \alpha^4/3 + \dots$; and $1 - \cos\alpha = \alpha^2/2 - \alpha^4/24$. We expand the effective potential (112) to quartic order about $\alpha = 0$,

$$V_{\text{eff}} \simeq -\frac{1}{2}ml^2\omega^2(\alpha^2 - \alpha^4/3) + \frac{1}{2}mgl(\alpha^2 - \alpha^4/12) = \frac{1}{2}ml^2[(g/l - \omega^2)\alpha^2 + (4\omega^2 - g/l)\alpha^4/12]. \quad (115)$$

The quartic term in the potential is positive for the range of ω^2 we are considering, $\omega^2 \gtrsim g/l$, so a sketch of the effective potential for two values of ω above and below ω_{crit} is as shown (dashed line is for $\omega > \omega_{\text{crit}}$).

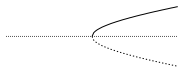
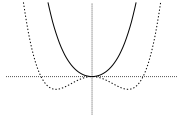
The new minima of the effective potential, for $\omega > \omega_{\text{crit}}$, can be found by differentiation.

$$dV_{\text{eff}}/d\alpha = ml^2[(\omega_{\text{crit}}^2 - \omega^2)\alpha + (4\omega^2 - \omega_{\text{crit}}^2)\alpha^3/6] \quad (116)$$

The slope is zero at $\alpha = 0$ and at the two points where

$$\alpha^2/6 = (\omega^2 - \omega_{\text{crit}}^2)/(4\omega^2 - \omega_{\text{crit}}^2). \quad (117)$$

For ω only a little greater than ω_{crit} , the variation of the new minimum with ω is as



$$\alpha \sim \pm \left(\frac{\omega - \omega_{\text{crit}}}{\omega_{\text{crit}}} \right)^{1/2}. \quad (118)$$

In fact, if we work out all the constants, we get:

$$\alpha \simeq \pm 6^{1/2} \left(\frac{2\omega - \omega_{\text{crit}}}{3\omega_{\text{crit}}} \right)^{1/2}, \quad (119)$$

but the details are not crucial to a sketch of the stable values of $\alpha^*(\omega)$.

The frequency of small oscillations about the new minimum is found from the second derivative, which we find by the hygienic method:

$$dV_{\text{rmeff}}/d\alpha = ml^2(\alpha)[(\omega_{\text{crit}}^2 - \omega^2) + (4\omega^2 - \omega_{\text{crit}}^2)\alpha^2/6] \quad (120)$$

so

$$d^2V_{\text{rmeff}}/d\alpha^2|_{\alpha^*} = ml^2(\alpha^*)((4\omega^2 - \omega_{\text{crit}}^2)\alpha^*/3) + 0 \quad (121)$$

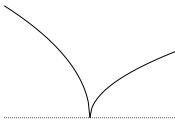
So

$$\Omega_{\text{SHM}}^2 \simeq (\alpha^*)^2/3((4\omega^2 - \omega_{\text{crit}}^2) \simeq (\omega^2 - \omega_{\text{crit}}^2)/3 \quad (122)$$

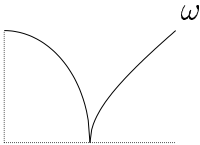
Compare this with the frequency below the critical point (114). So the frequency of these oscillations has the following behaviour: as ω approaches ω_{crit} , the frequency decreases to zero, as the square root of the distance from the critical point; then for $\omega > \omega_{\text{crit}}$, the frequency increases as the square root of the distance from the critical point, with an extra factor of 1/3, assuming I have made no slips. [I was actually expecting a factor of 1/2, so I should double-check this answer.]

The two sketches show Ω_{SHM} versus the rotation rate ω , first, in the neighbourhood of ω_{crit} , and second, the big picture, from $\omega = 0$ to $\omega = 2\omega_{\text{crit}}$. The frequency of small oscillations is equal to ω_{crit} when $\omega^2 \simeq 4\omega_{\text{crit}}^2$, i.e., $\omega \simeq 2\omega_{\text{crit}}$. [Actually, maybe we should include higher order corrections to be sure of the right answer to this last part.]

Ω_{SHM}



Ω_{SHM}



Elasticity

T22: The handout distributed October 2000 had a slip in the hint: the estimated k should be 16N/m.

Orbits

T24: Ellipses – The way to answer this sort of question is to identify all the constraints that the solution must satisfy:

1. the orbit, if it is a closed orbit in a $1/r$ potential, must be an ellipse *with the attractive origin at one focus*; [the most common error in these problems is to draw orbits that don't satisfy this constraint.]
2. if we get onto this orbit by receiving a kick at some point P , the orbit must come back through P ;
3. the tangent to the orbit at any point is in the same direction as the velocity at that point – in particular, if you know the velocity at P , you can deduce the tangent at P ;

FIGURES
NEEDED

4. ellipses have various useful properties, for example,

- (a) the ellipse is symmetrical about its major axis; of course, you may not be sure initially what the direction of the major axis is, but this fact is still a useful constraint.
- (b) there's only two points on an ellipse where the tangent is perpendicular to the radius vector – they are the two points lying on the major axis.
- (c) locally, you can think of an ellipse like the parabola along which a free-falling body falls – so sometimes you can use your knowledge of free-falling bodies to figure out the local picture.

Given what we discussed in lectures, you should be able to answer all parts of this question except for the final part, 'state the changes in period of the satellite' – we didn't discuss how period is related to the orbital parameters very much. However, if you remember the precise statement of Kepler's 3rd law, you should be able to do this bit too: K3 says that $T^2 \propto a^3$, where a is the length of the semi-major axis.

OK, let's solve the problems. First, let's do b , which we did in lecture. For small radial perturbations, the orbit can be thought of as an oscillation about the original circular orbit. The frequency of the oscillation happens to be the same as the original orbital frequency, so the perturbed orbit, by massive coincidence, is a closed orbit that wobbles inside and outside the circle once per orbit.

In case b , the energy is slightly increased by the kick (the new speed, post-kick, is slightly larger), and the period is increased (because the semi-major axis is a little larger); the angular momentum is unchanged.

In case d , we obtain an ellipse that is the mirror image of b . The energy is slightly increased, just as in b , the period is increased, and the angular momentum is unchanged.

Now comes the challenge: what about case a ? The kick is in the direction of the velocity, so the new velocity points in the same direction as before the kick. So the new ellipse must have a tangent in the same direction as the old circular orbit, at the kick point. The ellipse only has two points where its tangent is perpendicular to the radius vector – these are the closest and furthest distances from the attracting point, and they lie on the major axis. So the kick point, P, must lie on the major axis of the new ellipse. The new orbit is sketched in the figure. You can think of the motion close to P by analogy with the parabolic falling of a mass: the bigger the horizontal kick, the wider the parabola.

In case a , there is a significant increase in energy (bigger than in case b , because a kick along the direction of \mathbf{v} has a much bigger effect on the magnitude of \mathbf{v}). The angular momentum increases and the period increases.

Case c corresponds to the other orientation of a Kepler ellipse.